Contents 23

Fourier Series

Learning outcomes

In this Workbook you will learn how to express a periodic signal f(t) in a series of sines and cosines. You will learn how to simplify the calculations if the signal happens to be an even or an odd function. You will learn some brief facts relating to the convergence of the Fourier series. You will learn how to approximate a non-periodic signal by a Fourier series. You will learn how to re-express a standard Fourier series in complex form which paves the way for a later examination of Fourier transforms. Finally you will learn about some simple applications of Fourier series.

Periodic Functions

23.1

Introduction

You should already know how to take a function of a single variable $f(x)$ and represent it by a power series in x about any point x_0 of interest. Such a series is known as a Taylor series or Taylor expansion or, if $x_0 = 0$, as a Maclaurin series. This topic was firs met in HELM 16. Such an expansion is only possible if the function is sufficiently smooth (that is, if it can be differentiated as often as required). Geometrically this means that there are no *jumps* or *spikes* in the curve $y = f(x)$ near the point of expansion. However, in many practical situations the functions we have to deal with are not as well behaved as this and so no power series expansion in x is possible. Nevertheless, if the function is **periodic**, so that it repeats over and over again at regular intervals, then, irrespective of the function's behaviour (that is, no matter how many *jumps* or *spikes* it has), the function may be expressed as a series of sines and cosines. Such a series is called a **Fourier series**.

Fourier series have many applications in mathematics, in physics and in engineering. For example they are sometimes essential in solving problems (in heat conduction, wave propagation etc) that involve partial differential equations. Also, using Fourier series the analysis of many engineering systems (such as electric circuits or mechanical vibrating systems) can be extended from the case where the input to the system is a sinusoidal function to the more general case where the input is periodic but non-sinsusoidal.

Prerequisites

Before starting this Section you should . . .

- be familiar with trigonometric functions

Learning Outcomes

On completion you should be able to . . .

- recognise periodic functions
- determine the frequency, the amplitude and the period of a sinusoid
- represent common periodic functions by trigonometric Fourier series

1. Introduction

You have met in earlier Mathematics courses the concept of representing a function by an infinite series of simpler functions such as polynomials. For example, the Maclaurin series representing e^x has the form

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

or, in the more concise sigma notation,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(remembering that 0! is defined as 1).

The basic idea is that for those values of x for which the series converges we may approximate the function by using only the first few terms of the infinite series.

Fourier series are also usually infinite series but involve sine and cosine functions (or their complex exponential equivalents) rather than polynomials. They are widely used for approximating **periodic functions**. Such approximations are of considerable use in science and engineering. For example, elementary a.c. theory provides techniques for analyzing electrical circuits when the currents and voltages present are assumed to be sinusoidal. Fourier series enable us to extend such techniques to the situation where the functions (or signals) involved are periodic but not actually sinusoidal. You may also see in HELM 25 that Fourier series sometimes have to be used when solving partial differential equations.

2. Periodic functions

A function $f(t)$ is periodic if the function values repeat at regular intervals of the independent variable t. The regular interval is referred to as the **period**. See Figure 1.
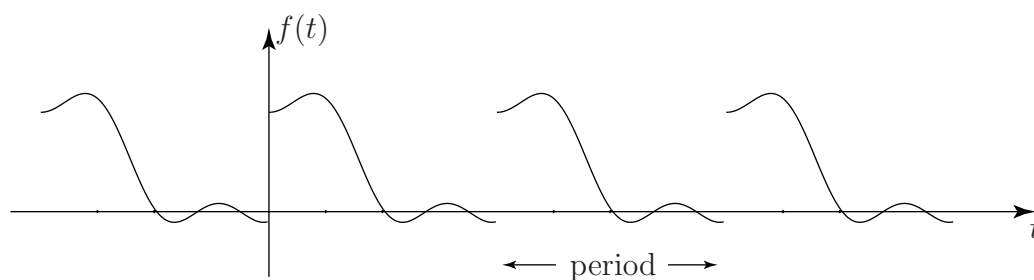

Figure 1

If P denotes the period we have

$$f(t + P) = f(t)$$

for any value of t.

The most obvious examples of periodic functions are the trigonometric functions $\sin t$ and $\cos t$, both of which have period 2π (using radian measure as we shall do throughout this Workbook) (Figure 2). This follows since

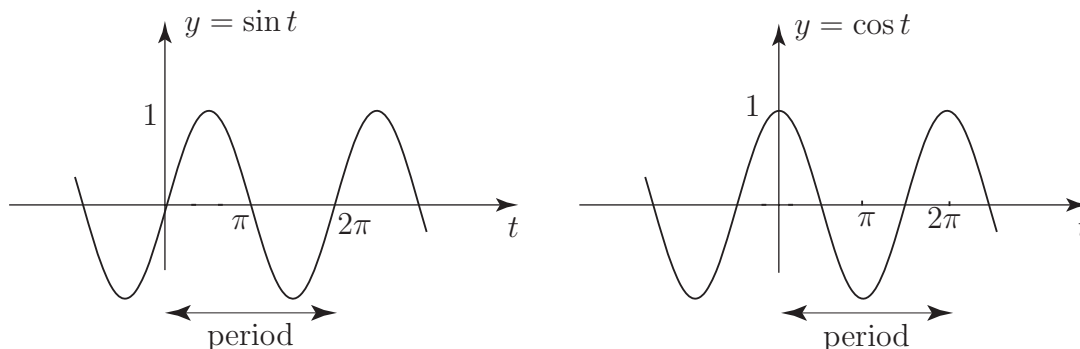$$\sin(t + 2\pi) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi) = \cos t$$

Figure 2

The **amplitude** of these sinusoidal functions is the maximum displacement from $y = 0$ and is clearly 1. (Note that we use the term sinusoidal to include cosine as well as sine functions.)

More generally we can consider a sinusoid

$$y = A \sin nt$$

which has maximum value, or amplitude, A and where n is usually a positive integer.

For example

$$y = \sin 2t$$

is a sinusoid of amplitude 1 and period $\dfrac{2\pi}{2} = \pi$ (Figure 3). The fact that the period is π follows because

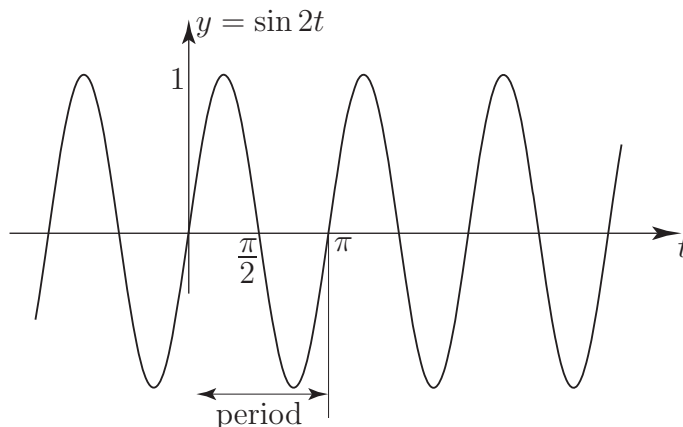$$\sin 2(t + \pi) = \sin(2t + 2\pi) = \sin 2t$$

for any value of t.

Figure 3

We see that $y = \sin 2t$ has half the period of $\sin t$, π as opposed to 2π (Figure 4). This can alternatively be phrased by stating that $\sin 2t$ oscillates twice as rapidly (or has twice the frequency) of $\sin t$.
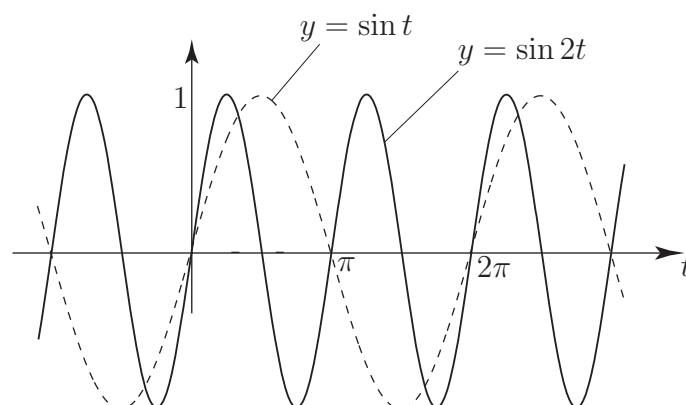

Figure 4

In general $y = A \sin nt$ has amplitude A, period $\dfrac{2\pi}{n}$ and completes n oscillations when t changes by 2π. Formally, we define the **frequency** of a sinusoid as the reciprocal of the period:

$$\text{frequency} = \frac{1}{\text{period}}$$

and the **angular frequency**, often denoted the Greek Letter ω (omega) as

$$\text{angular frequency} = 2\pi \times \text{frequency} = \frac{2\pi}{\text{period}}$$

Thus $y = A \sin nt$ has frequency $\dfrac{n}{2\pi}$ and angular frequency n.

Task

State the amplitude, period, frequency and angular frequency of

(a) $y = 5 \cos 4t$ (b) $y = 6 \sin \dfrac{2t}{3}$.

Your solution

(a)

Answer

amplitude 5, period $\dfrac{2\pi}{4} = \dfrac{\pi}{2}$, frequency $\dfrac{2}{\pi}$, angular frequency 4

Your solution

(b)

Answer

amplitude 6, period 3π, frequency $\dfrac{1}{3\pi}$, angular frequency $\dfrac{2}{3}$

Harmonics

In representing a non-sinusoidal function of period 2π by a Fourier series we shall see shortly that only certain sinusoids will be required:

(a) $A_1 \cos t$ (and $B_1 \sin t$)

These also have period 2π and together are referred to as the **first harmonic** (or **fundamental harmonic**).

(b) $A_2 \cos 2t$ (and $B_2 \sin 2t$)

These have half the period, and double the frequency, of the first harmonic and are referred to as the **second harmonic**.

(c) $A_3 \cos 3t$ (and $B_3 \sin 3t$)

These have period $\dfrac{2\pi}{3}$ and constitute the **third harmonic**.

In general the Fourier series of a function of period 2π will require harmonics of the type

$$A_n \cos nt \qquad (\text{ and } \quad B_n \sin nt) \qquad \text{where} \quad n = 1, 2, 3, \ldots$$

Non-sinusoidal periodic functions

The following are examples of non-sinusoidal periodic functions (they are often called "waves").
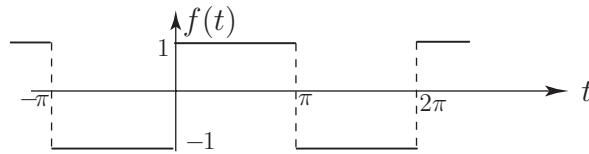
Square wave

Figure 5

Analytically we can describe this function as follows:

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ +1 & 0 < t < \pi \end{cases} \qquad \text{(which gives the definition over one period)}$$

$$f(t + 2\pi) = f(t) \qquad \text{(which tells us that the function has period } 2\pi\text{)}$$
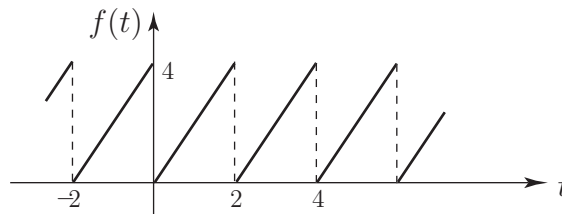
Saw-tooth wave

Figure 6

In this case we can describe the function as follows:

$$f(t) = 2t \qquad 0 < t < 2 \qquad f(t + 2) = f(t)$$

Here the period is 2, the frequency is $\dfrac{1}{2}$ and the angular frequency is $\dfrac{2\pi}{2} = \pi$.

Triangular wave
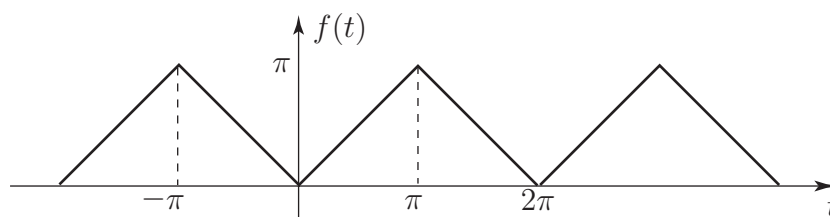
Figure 7

Here we can conveniently define the function using $-\pi < t < \pi$ as the "basic period":

$$f(t) = \begin{cases} -t & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases}$$

or, more concisely,

$$f(t) = |t| \qquad -\pi < t < \pi$$

together with the usual statement on periodicity

$$f(t + 2\pi) = f(t).$$

Task

Write down an analytic definition for the following periodic function:
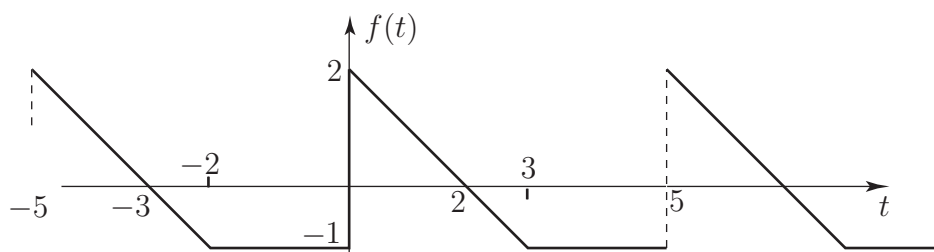
Your solution

Answer

$$f(t) = \begin{cases} 2 - t & 0 < t < 3 \\ -1 & 3 < t < 5 \end{cases} \qquad\qquad f(t + 5) = f(t)$$

Task

Sketch the graphs of the following periodic functions showing all relevant values:

(a) $\quad f(t) = \begin{cases} t^2/2 & 0 < t < 4 \\ 8 & 4 < t < 6 \\ 0 & 6 < t < 8 \end{cases} \qquad\qquad f(t+8) = f(t)$

(b) $\quad f(t) = 2t - t^2 \qquad 0 < t < 2 \qquad\qquad f(t+2) = f(t)$

Your solution

Answer

Figure 9

Representing Periodic Functions by Fourier Series
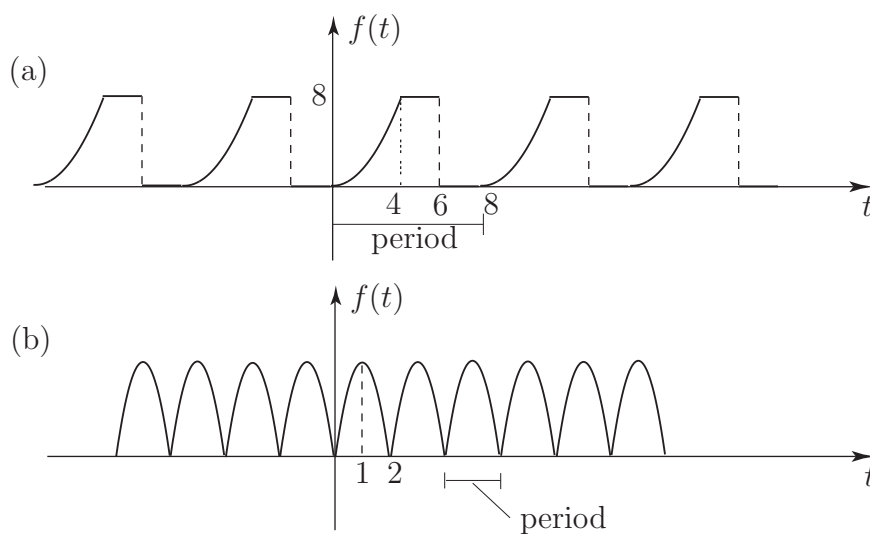
23.2

⚡ Introduction

In this Section we show how a periodic function can be expressed as a series of sines and cosines. We begin by obtaining some standard integrals involving sinusoids. We then **assume** that if $f(t)$ is a periodic function, of period 2π, then the Fourier series expansion takes the form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Our main purpose here is to show how the constants in this expansion, a_n (for $n = 0, 1, 2, 3 \ldots$ and b_n (for $n = 1, 2, 3, \ldots$), may be determined for any given function $f(t)$.

⚡ **Prerequisites** Before starting this Section you should ...	• know what a periodic function is • be able to integrate functions involving sinusoids • have knowledge of integration by parts
⚡ **Learning Outcomes** On completion you should be able to ...	• calculate Fourier coefficients of a function of period 2π • calculate Fourier coefficients of a function of general period

1. Introduction

We recall first a simple trigonometric identity:

$$\cos 2t = -1 + 2\cos^2 t \qquad \text{or equivalently} \qquad \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t \tag{1}$$

Equation 1 can be interpreted as a simple **finite** Fourier series representation of the periodic function $f(t) = \cos^2 t$ which has period π. We note that the Fourier series representation contains a constant term and a period π term.

A more complicated trigonometric identity is

$$\sin^4 t = \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t \tag{2}$$

which again can be considered as a finite Fourier series representation. (Do not worry if you are unfamiliar with the result (2).) Note that the function $f(t) = \sin^4 t$ (which has period π) is being written in terms of a constant function, a function of period π or frequency $\dfrac{1}{\pi}$ (the "first harmonic") and a function of period $\dfrac{\pi}{2}$ or frequency $\dfrac{2}{\pi}$ (the "second harmonic").

The reason for the constant term in both (1) and (2) is that each of the functions $\cos^2 t$ and $\sin^4 t$ is non-negative and hence each must have a positive average value. Any sinusoid of the form $\cos nt$ or $\sin nt$ has, by symmetry, zero average value. Therefore, so would a Fourier series containing only such terms. A constant term can therefore be expected to arise in the Fourier series of a function which has a non-zero average value.

2. Functions of period 2π

We now discuss how to represent periodic non-sinusoidal functions $f(t)$ of period 2π in terms of sinusoids, i.e. how to obtain Fourier series representations. As already discussed we expect such Fourier series to contain harmonics of frequency $\dfrac{n}{2\pi}$ ($n = 1, 2, 3, \ldots$) and, if the periodic function has a non-zero average value, a constant term.

Thus we seek a Fourier series representation of the general form

$$f(t) = \frac{a_0}{2} + a_1\cos t + a_2\cos 2t + \ldots + b_1\sin t + b_2\sin 2t + \ldots$$

The reason for labelling the constant term as $\dfrac{a_0}{2}$ will be discussed later. The amplitudes $a_1, a_2, \ldots$ $b_1, b_2, \ldots$ of the sinusoids are called **Fourier coefficients**.

Obtaining the Fourier coefficients for a given periodic function $f(t)$ is our main task and is referred to as Fourier Analysis. Before embarking on such an analysis it is instructive to establish, at least qualitatively, the plausibility of approximating a function by a few terms of its Fourier series.

Task

Consider the square wave of period 2π one period of which is shown in Figure 10.
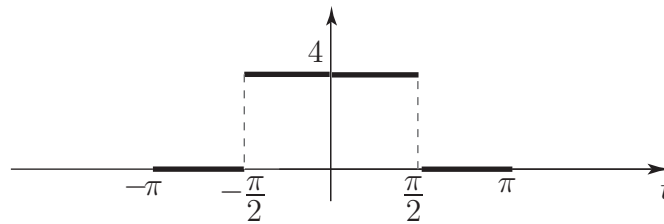
(a) Write down the analytic description of this function,

(b) State whether you expect the Fourier series of this function to contain a constant term,

(c) List any other possible features of the Fourier series that you might expect from the graph of the square-wave function.

Your solution

Answer

(a) We have

$$f(t) = \begin{cases} 4 & -\dfrac{\pi}{2} < t < \dfrac{\pi}{2} \\[2mm] 0 & -\pi < t < -\dfrac{\pi}{2}, \ \dfrac{\pi}{2} < t < \pi \end{cases}$$

$$f(t + 2\pi) = f(t)$$

(b) The Fourier series will contain a constant term since the square wave here is non-negative and cannot therefore have a zero average value. This constant term is often referred to as the d.c. (direct current) term by engineers.

(c) Since the square wave is an even function (i.e. the graph has symmetry about the y axis) then its Fourier series will contain cosine terms but not sine terms because only the cosines are even functions. (Well done if you spotted this at this early stage!)

It is possible to show, and we will do so later, that the Fourier series representation of this square wave is

$$2 + \frac{8}{\pi}\left\{\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \frac{1}{7}\cos 7t + \dots\right\}$$

i.e. the Fourier coefficients are

$$\frac{a_0}{2} = 2, \qquad a_1 = \frac{8}{\pi}, \qquad a_2 = 0, \qquad a_3 = -\frac{8}{3\pi}, \qquad a_4 = 0, \qquad a_5 = \frac{8}{5\pi}, \dots$$

Note, as well as the presence of the constant term and of the cosine (but not sine) terms, that only odd harmonics are present i.e. sinusoids of period 2π, $\frac{2\pi}{3}$, $\frac{2\pi}{5}$, $\frac{2\pi}{7}$, ... or of frequency $1, 3, 5, 7, \dots$ times the fundamental frequency $\frac{1}{2\pi}$.

We now show in Figure 8 graphs of

(i) the square wave

(ii) the first two terms of the Fourier series representing the square wave

(iii) the first three terms of the Fourier series representing the square wave

(iv) the first four terms of the Fourier series representing the square wave

(v) the first five terms of the Fourier series representing the square wave

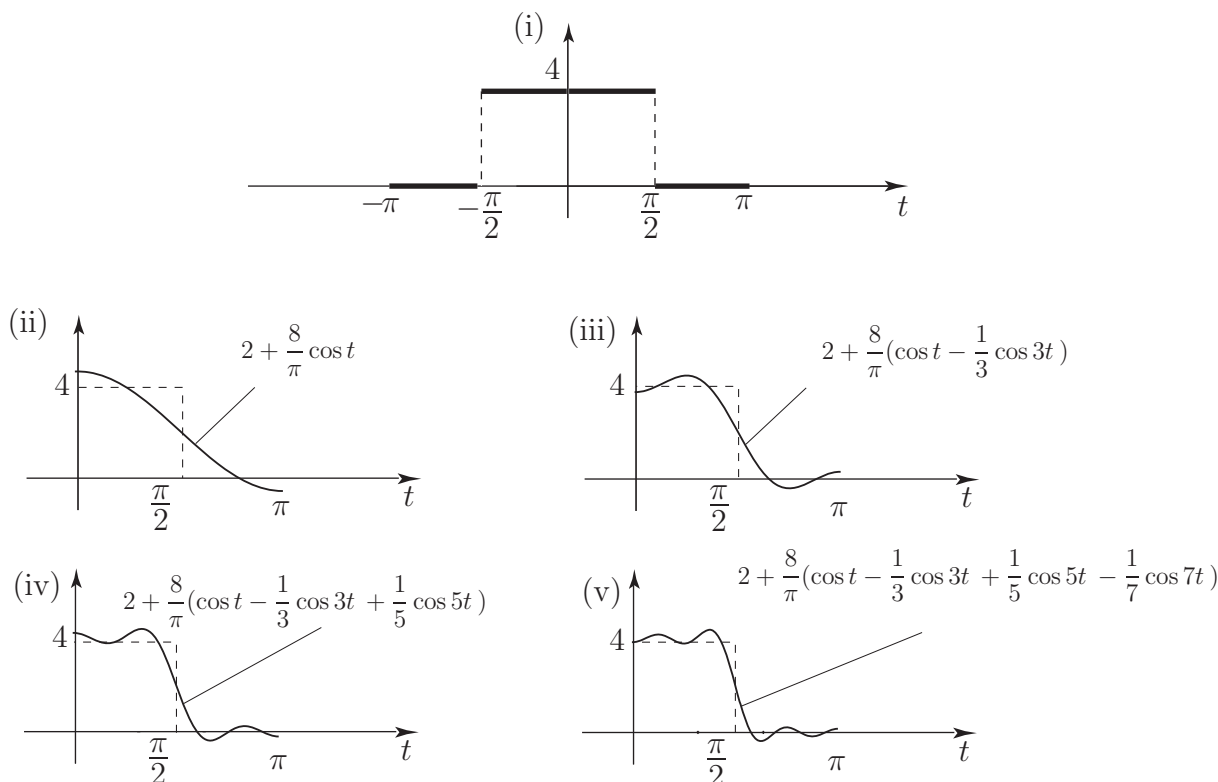Note: We show the graphs for $0 < t < \pi$ only since the square wave and its Fourier series are even.

Figure 8

We can clearly see from Figure 8 that as the number of terms is increased the graph of the Fourier series gradually approaches that of the original square wave - the ripples increase in number but decrease in amplitude. (The behaviour near the **discontinuity**, at $t = \dfrac{\pi}{2}$, is slightly more complicated and it is possible to show that however many terms are taken in the Fourier series, some "overshoot" will always occur. This effect, which we do not discuss further, is known as the Gibbs Phenomenon.)

Orthogonality properties of sinusoids

As stated earlier, a periodic function $f(t)$ with period 2π has a Fourier series representation

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + a_1 \cos t + a_2 \cos 2t + \ldots + b_1 \sin t + b_2 \sin 2t + \ldots, \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{3}
\end{aligned}
$$

To determine the Fourier coefficients a_n, b_n and the constant term $\dfrac{a_0}{2}$ use has to be made of certain integrals involving sinusoids, the integrals being over a range $\alpha, \alpha + 2\pi$, where α is any number. (We will normally choose $\alpha = -\pi$.)

Task

Find $\displaystyle\int_{-\pi}^{\pi} \sin nt \, dt$ and $\displaystyle\int_{-\pi}^{\pi} \cos nt \, dt$ where n is an integer.

Your solution

Answer

In fact both integrals are zero for

$$
\int_{-\pi}^{\pi} \sin nt \, dt = \left[-\frac{1}{n} \cos nt \right]_{-\pi}^{\pi} = \frac{1}{n} \left(-\cos n\pi + \cos n\pi \right) = 0 \qquad n \neq 0 \tag{4}
$$

$$
\int_{-\pi}^{\pi} \cos nt \, dt = \left[\frac{1}{n} \sin nt \right]_{-\pi}^{\pi} = 0 \qquad n \neq 0 \tag{5}
$$

As special cases, if $n = 0$ the first integral is zero and the second integral has value 2π.

N.B. Any integration range $\alpha, \alpha + 2\pi$, would give these same (zero) answers.

These integrals enable us to calculate the constant term in the Fourier series (3) as in the following task.

Task

Integrate both sides of (3) from $-\pi$ to π and use the results from the previous Task. Hence obtain an expression for a_0.

Your solution

Answer

We get for the left-hand side

$$\int_{-\pi}^{\pi} f(t)\,dt$$

(whose value clearly depends on the function $f(t)$).

Integrating the right-hand side term by term we get

$$\frac{1}{2}\int_{-\pi}^{\pi} a_0\ dt + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nt\ dt + \int_{-\pi}^{\pi} b_n \sin nt\ dt\right\} = \frac{1}{2}\left[a_0\ t\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\{0 + 0\}$$

(using the integrals (4) and (5) shown above). Thus we get

$$\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2}(2a_0\pi) \qquad \text{or} \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \tag{6}$$

Key Point 1

The constant term in a trigonometric Fourier series for a function of period 2π is

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \text{ average value of } f(t) \text{ over 1 period.}$$

This result ties in with our earlier discussion on the significance of the constant term. Clearly a signal whose average value is zero will have no constant term in its Fourier series. The following square wave (Figure 9) is an example.
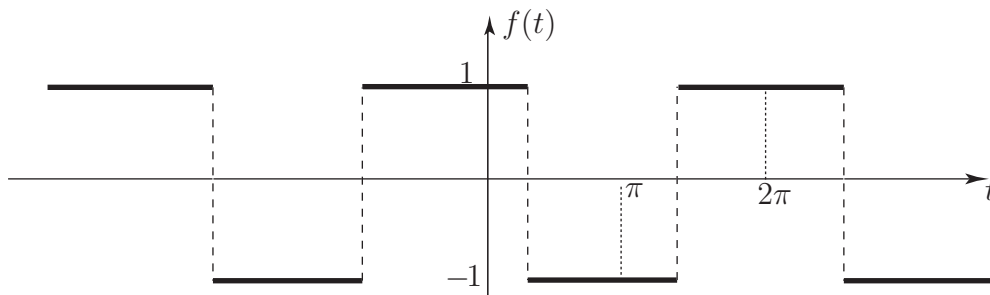

Figure 9

We now obtain further integrals, known as orthogonality properties, which enable us to find the remaining Fourier coefficients i.e. the amplitudes a_n and b_n $(n = 1, 2, 3, \ldots)$ of the sinusoids.

Task

Using the standard trigonometric identity that

$$\sin nt \cos mt \equiv \frac{1}{2} \left\{ \sin(n+m)t + \sin(n-m)t \right\}$$

evaluate $\displaystyle\int_{-\pi}^{\pi} \sin nt \cos mt \ dt$ where n and m are any integers.

Your solution

Answer
We get

$$\int_{-\pi}^{\pi} \sin nt \cos mt \ dt = \frac{1}{2} \left\{ \int_{-\pi}^{\pi} \sin(n+m)t \ dt + \int_{-\pi}^{\pi} \sin(n-m)t \ dt \right\} = \frac{1}{2}\{0 + 0\} = 0$$

using the results (4) and (5) since $n + m$ and $n - m$ are also integers.

This result holds for any interval of 2π.

Key Point 2

Orthogonality Relation

For any integers m, n, including the case $m = n$,

$$\int_{-\pi}^{\pi} \sin nt \cos mt \ dt = 0$$

We shall use this result shortly but need a few more integrals first.
Consider next

$$\int_{-\pi}^{\pi} \cos nt \cos mt \ dt \qquad \text{where } m \text{ and } n \text{ are integers.}$$

Using another trigonometric identity we have, for the case $n \neq m$,

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos nt \cos mt \ dt &= \frac{1}{2} \int_{-\pi}^{\pi} \{\cos(n + m)t + \cos(n - m)t\} dt \\
&= \frac{1}{2}\{0 + 0\} = 0 \quad \text{using the integrals (4) and (5).}
\end{aligned}
$$

For the case $n = m$ we must get a non-zero answer since $\cos^2 nt$ is non-negative. In this case:

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos^2 nt \ dt &= \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2nt) \, dt \\
&= \frac{1}{2} \left[t + \frac{1}{2n} \sin 2nt \right]_{-\pi}^{\pi} = \pi \qquad (\text{ provided } n \neq 0)
\end{aligned}
$$

For the case $n = m = 0$ we have $\quad \int_{-\pi}^{\pi} \cos nt \cos mt \ dt = 2\pi$

Task

Proceeding in a similar way to the above, evaluate

$$\int_{-\pi}^{\pi} \sin nt \sin mt \ \ dt$$

for integers m and n.

Again consider separately the three cases: (a) $n \neq m$, (b) $n = m \neq 0$ and (c) $n = m = 0$.

Your solution

Answer

(a) Using the identity $\sin nt \sin mt \equiv \frac{1}{2}\{\cos(n-m)t - \cos(n+m)t\}$ and integrating the right-hand side terms, we get, using (4) and (5)

$$\int_{-\pi}^{\pi} \sin nt \sin mt \, dt = 0 \qquad n, m \text{ integers} \qquad n \neq m$$

(b) Using the identity $\cos 2\theta = 1 - 2\sin^2\theta$ with $\theta = nt$ gives for $n = m \neq 0$

$$\int_{-\pi}^{\pi} \sin^2 nt \, dt \;=\; \frac{1}{2}\int_{-\pi}^{\pi}(1 - \cos 2nt)dt = \pi$$

(c) When $n = m = 0$, $\int_{-\pi}^{\pi} \sin nt \sin mt \, dt = 0$.

We summarise these results in the following Key Point:

Key Point 3

For integers n, m

$$\int_{-\pi}^{\pi} \sin nt \cos mt \, dt \;=\; 0$$

$$\int_{-\pi}^{\pi} \cos nt \cos mt \, dt \;=\; \begin{cases} 0 & n \neq m \\ \pi & n = m \neq 0 \\ 2\pi & n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nt \sin mt \, dt \;=\; \begin{cases} 0 & n \neq m, \; n = m = 0 \\ \pi & n = m \end{cases}$$

All these results hold for any integration range of width 2π.

3. Calculation of Fourier coefficients

Consider the Fourier series for a function $f(t)$ of period 2π:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{7}$$

To obtain the coefficients a_n $(n = 1, 2, 3, \ldots)$, we multiply both sides by $\cos mt$ where m is some positive integer and integrate both sides from $-\pi$ to π.

For the left-hand side we obtain

$$\int_{-\pi}^{\pi} f(t) \cos mt \, dt$$

For the right-hand side we obtain

$$\frac{a_0}{2}\int_{-\pi}^{\pi}\cos mt \, dt + \sum_{n=1}^{\infty}\left\{ a_n \int_{-\pi}^{\pi}\cos nt \cos mt \, dt + b_n \int_{-\pi}^{\pi}\sin nt \cos mt \, dt \right\}$$

The first integral is zero using (5).

Using the orthogonality relations all the integrals in the summation give zero except for the case $n = m$ when, from Key Point 3

$$\int_{-\pi}^{\pi} \cos^2 mt \, dt = \pi$$

Hence

$$\int_{-\pi}^{\pi} f(t) \cos mt \, dt = a_m \pi$$

from which the coefficient a_m can be obtained.

Rewriting m as n we get

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos nt \, dt \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{8}$$

Using (6), we see the formula also works for $n = 0$ (but we must remember that the constant term is $\frac{a_0}{2}$.)

From (8)

$\qquad a_n = 2 \times$ average value of $f(t)\cos nt$ over one period.

Task

By multiplying (7) by $\sin mt$ obtain an expression for the Fourier Sine coefficients b_n, $n = 1, 2, 3, \ldots$

Your solution

Answer

A similar calculation to that performed to find the a_n gives

$$\int_{-\pi}^{\pi} f(t) \sin mt \, dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin mt \, dt + \sum_{n=1}^{\infty} \left\{ \int_{-\pi}^{\pi} a_n \cos nt \sin mt \, dt + \int_{-\pi}^{\pi} b_n \sin nt \sin mt \, dt \right\}$$

All terms on the right-hand side integrate to zero except for the case $n = m$ where

$$\int_{-\pi}^{\pi} b_m \sin^2 mt \, dt = b_m \pi$$

Relabelling m as n gives

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \qquad n = 1, 2, 3, \ldots \tag{9}$$

(There is no Fourier coefficient b_0.)

Clearly $b_n = 2 \times$ average value of $f(t) \sin nt$ over one period.

Key Point 4

A function $f(t)$ with period 2π has a Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

The Fourier coefficients are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt \qquad n = 1, 2, \ldots$$

In the integrals any convenient integration range extending over an interval of 2π may be used.

4. Examples of Fourier series

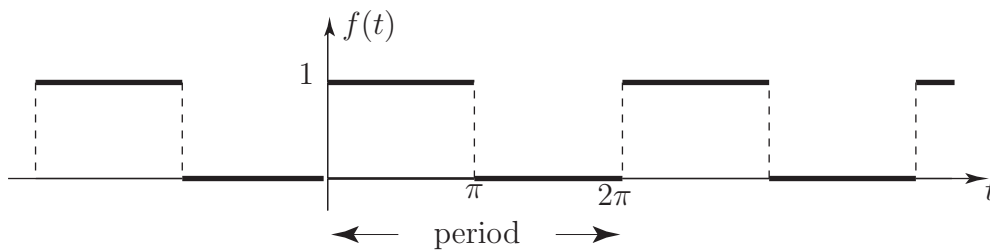We shall obtain the Fourier series of the "half-rectified" square wave shown in Figure 10.

Figure 10

We have

$$f(t) = \begin{cases} 1 & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

$$f(t + 2\pi) = f(t)$$

The calculation of the Fourier coefficients is merely straightforward integration using the results already obtained:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt$$

in general. Hence, for our square wave

$$a_n = \frac{1}{\pi} \int_{0}^{\pi} (1) \cos nt\, dt = \frac{1}{\pi} \left[\frac{\sin nt}{n} \right]_{0}^{\pi} = 0 \qquad \text{provided } n \neq 0$$

But $a_0 = \dfrac{1}{\pi} \displaystyle\int_0^\pi (1)\, dt = 1$ so the constant term is $\dfrac{a_0}{2} = \dfrac{1}{2}$.

(The square wave takes on values 1 and 0 over equal length intervals of t so $\dfrac{1}{2}$ is clearly the mean value.)

Similarly

$$b_n = \frac{1}{\pi} \int_0^\pi (1) \sin nt\, dt = \frac{1}{\pi} \left[-\frac{\cos nt}{n} \right]_0^\pi$$

Some care is needed now!

$$b_n = \frac{1}{n\pi} \left(1 - \cos n\pi \right)$$

But $\quad \cos n\pi = +1 \qquad n = 2, 4, 6, \ldots,$

$$\therefore \qquad b_n = 0 \qquad n = 2, 4, 6, \ldots$$

However, $\quad \cos n\pi = -1 \qquad n = 1, 3, 5, \ldots$

$$\therefore \qquad b_n = \frac{1}{n\pi}(1 - (-1)) = \frac{2}{n\pi} \qquad n = 1, 3, 5, \ldots$$

i.e. $\quad b_1 = \dfrac{2}{\pi}, \; b_3 = \dfrac{2}{3\pi}, \; b_5 = \dfrac{2}{5\pi}, \ldots$

Hence the required Fourier series is

$$f(t) \;=\; \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad \text{in general}$$

$$f(t) \;=\; \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \ldots \right) \qquad \text{in this case}$$

Note that the Fourier series for this particular form of the square wave contains a constant term and odd harmonic sine terms. We already know why the constant term arises (because of the non-zero mean value of the functions) and will explain later why the presence of any odd harmonic sine terms could have been predicted without integration.
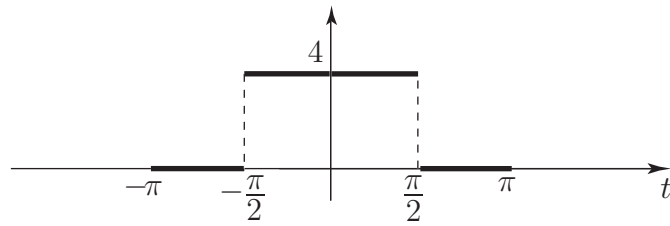
The Fourier series we have found can be written in summation notation in various ways:

$\dfrac{1}{2} + \dfrac{2}{\pi} \displaystyle\sum_{\substack{n=1 \\ (n\ \text{odd})}}^{\infty} \dfrac{1}{n} \sin nt \quad$ or, since n is odd, we may write $\quad n = 2k - 1 \quad k = 1, 2, \ldots$ and write the

Fourier series as $\quad \dfrac{1}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{(2k-1)} \sin(2k-1)t$

Task

Obtain the Fourier series of the square wave one period of which is shown:

Your solution

Answer

We have, since the function is non-zero only for $-\frac{\pi}{2} < t < \frac{\pi}{2}$,

$$a_0 = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \, dt = 4$$

$\therefore \quad \dfrac{a_0}{2} = 2$ is the constant term as we would expect. Also

$$
\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cos nt \, dt = \frac{4}{\pi} \left[\frac{\sin nt}{n} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{4}{n\pi} \left\{ \sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right) \right\} = \frac{8}{n\pi} \sin\left(\frac{n\pi}{2}\right) \qquad n = 1, 2, 3, \ldots
\end{aligned}
$$

It follows from a knowledge of the sine function that

$$
a_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\[2mm] \dfrac{8}{n\pi} & n = 1, 5, 9, \ldots \\[2mm] -\dfrac{8}{n\pi} & n = 3, 7, 11, \ldots \end{cases}
$$

Also

$$b_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \sin nt \, dt = \frac{4}{\pi} \left[-\frac{\cos nt}{n} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{4}{n\pi} \left\{ \cos\left(\frac{n\pi}{2}\right) - \cos\left(-\frac{n\pi}{2}\right) \right\} = 0$$

Hence, the required Fourier series is

$$f(t) = 2 + \frac{8}{\pi} \left(\cos t - \frac{1}{3} \cos 3t + \frac{1}{5} \cos 5t - \frac{1}{7} \cos 7t + \ldots \right)$$

which, like the previous square wave, contains a constant term and odd harmonics, but in this case odd harmonic cosine terms rather than sine.

You may recall that this particular square wave was used earlier and we have already sketched the form of the Fourier series for 2, 3, 4 and 5 terms in Figure 8.

Clearly, in finding the Fourier series of square waves, the integration is particularly simple because $f(t)$ takes on piecewise constant values. For other functions, such as saw-tooth waves this will not be the case. Before we tackle such functions however we shall generalise our formulae for the Fourier coefficients a_n, b_n to the case of a periodic function of arbitrary period, rather than confining ourselves to period 2π.

5. Fourier series for functions of general period

This is a straightforward extension of the period 2π case that we have already discussed.

Using x (instead of t) temporarily as the variable. We have seen that a 2π periodic function $f(x)$ has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx \qquad n = 0,1,2,\ldots \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx \qquad n = 1,2,\ldots$$

Suppose we now change the variable to t where $x = \dfrac{2\pi}{T}t$.

Thus $x = \pi$ corresponds to $t = T/2$ and $x = -\pi$ corresponds to $t = -T/2$.
Hence regarded as a function of t, we have a function with period T.

Making the substitution $x = \dfrac{2\pi}{T}t$, and hence $dx = \dfrac{2\pi}{T}dt$, in the expressions for a_n and b_n we obtain

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos\left(\frac{2n\pi t}{T}\right) dt \qquad n = 0,1,2\ldots$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin\left(\frac{2n\pi t}{T}\right) dt \qquad n = 1,2\ldots$$

These integrals give the Fourier coefficients for a function of period T whose Fourier series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right\}$$

Various other notations are commonly used in this case e.g. it is sometimes convenient to write the period $T = 2\ell$. (This is particularly useful when Fourier series arise in the solution of partial differential equations.) Another alternative is to use the angular frequency ω and put $T = 2\pi/\omega$.

Task

Write down the form of the Fourier series and expressions for the coefficients if
(a) $T = 2\ell$ (b) $T = 2\pi/\omega$.

Your solution

Answer

(a) $f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \left\{ a_n \cos\left(\dfrac{n\pi t}{\ell}\right) + b_n \sin\left(\dfrac{n\pi t}{\ell}\right) \right\}$ with $a_n = \dfrac{1}{\ell} \displaystyle\int_{-\ell}^{\ell} f(t) \cos\left(\dfrac{n\pi t}{\ell}\right) dt$

and similarly for b_n.

(b) $f(t) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \left\{ a_n \cos(n\omega t) + b_n \sin(n\omega t) \right\}$ with $a_n = \dfrac{\omega}{\pi} \displaystyle\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t) \cos(n\omega t) dt$

and similarly for b_n.

You should note that, as usual, any convenient integration range of length T (or 2ℓ or $\dfrac{2\pi}{\omega}$) can be used in evaluating a_n and b_n.

Example 1

Find the Fourier series of the function shown in Figure 11 which is a saw-tooth wave with alternate portions removed.
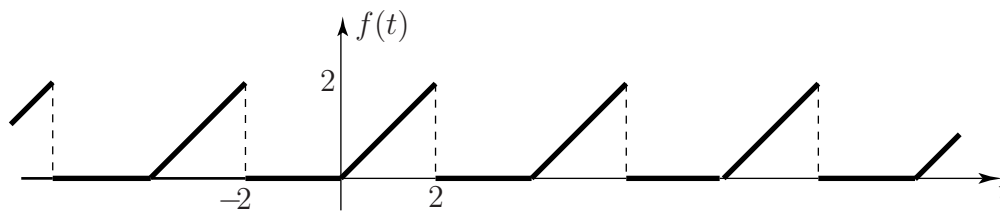

Figure 11

Solution

Here the period $T = 2\ell = 4$ so $\ell = 2$. The Fourier series will have the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi t}{2}\right) + b_n \sin\left(\frac{n\pi t}{2}\right) \right\}$$

The coefficients a_n are given by

$$a_n = \frac{1}{2} \int_{-2}^{2} f(t) \cos\left(\frac{n\pi t}{2}\right) dt$$

where

$$f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases} \qquad f(t+4) = f(t)$$

Hence $a_n = \dfrac{1}{2} \displaystyle\int_{0}^{2} t \cos\left(\dfrac{n\pi t}{2}\right) dt$.

Solution (contd.)

The integration is readily performed using integration by parts:

$$
\int_0^2 t \cos\left(\frac{n\pi t}{2}\right)\, dt \;=\; \left[t\,\frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right) \right]_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi t}{2}\right)\, dt
$$

$$
=\; \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi t}{2}\right) \right]_0^2 \qquad n \neq 0
$$

$$
=\; \frac{4}{n^2\pi^2}(\cos n\pi - 1).
$$

Hence, since $a_n = \dfrac{1}{2}\displaystyle\int_0^2 t\cos(\frac{n\pi t}{2})dt$

$$
a_n = \begin{cases} \quad 0 & n = 2, 4, 6, \ldots \\[2mm] -\dfrac{4}{n^2\pi^2} & n = 1, 3, 5, \ldots \end{cases}
$$

The constant term is $\dfrac{a_0}{2}$ where $\quad a_0 = \dfrac{1}{2}\displaystyle\int_0^2 t\, dt = 1.$

Similarly

$$
b_n = \frac{1}{2}\int_0^2 t\sin\left(\frac{n\pi t}{2}\right)\, dt
$$

where

$$
\int_0^2 t\sin\left(\frac{n\pi t}{2}\right)\, dt = \left[-t\frac{2}{n\pi}\cos\left(\frac{n\pi t}{2}\right) \right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi t}{2}\right)\, dt.
$$

The second integral gives zero. Hence

$$
b_n = -\frac{2}{n\pi}\cos n\pi = \begin{cases} -\dfrac{2}{n\pi} & n = 2, 4, 6, \ldots \\[3mm] +\dfrac{2}{n\pi} & n = 1, 3, 5, \ldots \end{cases}
$$

Hence, using all these results for the Fourier coefficients, the required Fourier series is

$$
f(t) = \frac{1}{2} \;-\; \frac{4}{\pi^2}\left\{ \cos\left(\frac{\pi t}{2}\right) + \frac{1}{9}\cos\left(\frac{3\pi t}{2}\right) + \frac{1}{25}\cos\left(\frac{5\pi t}{2}\right) + \ldots \right\}
$$

$$
+\; \frac{2}{\pi}\left\{ \sin\left(\frac{\pi t}{2}\right) - \frac{1}{2}\sin\left(\frac{2\pi t}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi t}{2}\right) \ldots \right\}
$$

Notice that because the Fourier coefficients depend on $\dfrac{1}{n^2}$ (rather than $\dfrac{1}{n}$ as was the case for the square wave) the sinusoidal components in the Fourier series have quite rapidly decreasing amplitudes. We would therefore expect to be able to approximate the original saw-tooth function using only a quite small number of terms in the series.

Task

Obtain the Fourier series of the function

$$f(t) = t^2 \qquad -1 < t < 1$$
$$f(t+2) = f(t)$$
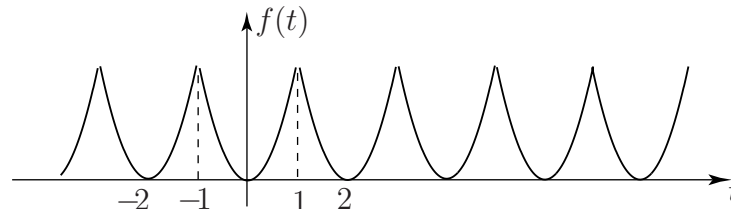
First write out the form of the Fourier series in this case:

Your solution

Answer
Since $T = 2\ell = 2$ and since the function has a non-zero average value, the form of the Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \{a_n(\cos n\pi t) + b_n \sin(n\pi t)\}$$

Now write out integral expressions for a_n and b_n. Will there be a constant term in the Fourier series?

Your solution

Answer
Because the function is non-negative there will be a constant term. Since $T = 2\ell = 2$ then $\ell = 1$ and we have

$$a_n = \int_{-1}^{1} t^2 \cos(n\pi t)\, dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \int_{-1}^{1} t^2 \sin(n\pi t)\, dt \qquad n = 1, 2, \ldots$$

The constant term will be $\dfrac{a_0}{2}$ where $a_0 = \displaystyle\int_{-1}^{1} t^2\, dt$.

Now evaluate the integrals. Try to spot the value of the integral for b_n so as to avoid integration. Note that the integrand is an even functions for a_n and an odd functon for b_n.

Your solution

Answer
The integral for b_n is zero for all n because the integrand is an odd function of t. Since the integrand is even in the integrals for a_n we can write

$$a_n = 2 \int_0^1 t^2 \cos n\pi t \, dt \qquad n = 0, 1, 2, \ldots$$

The constant term will be $\dfrac{a_o}{2}$ where $\quad a_0 = 2 \int_0^1 t^2 \, dt = \dfrac{2}{3}.$

For $n = 1, 2, 3, \ldots$ we must integrate by parts (twice)

$$
\begin{aligned}
a_n &= 2 \left\{ \left[\frac{t^2}{n\pi} \sin(n\pi t) \right]_0^1 - \frac{2}{n\pi} \int_0^1 t \sin(n\pi t) \, dt \right\} \\
&= -\frac{4}{n\pi} \left\{ \left[-\frac{t}{n\pi} \cos(n\pi t) \right]_0^1 + \frac{1}{n\pi} \int_0^1 \cos(n\pi t) \, dt \right\}.
\end{aligned}
$$

The integral in the second term gives zero so $\quad a_n = \dfrac{4}{n^2\pi^2} \cos n\pi.$

Now writing out the final form of the Fourier series we have

$$f(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2} \cos(n\pi t) = \frac{1}{3} + \frac{4}{\pi^2} \left\{ -\cos(\pi t) + \frac{1}{4}\cos(2\pi t) - \frac{1}{9}\cos(3\pi t) + \ldots \right\}$$

Exercises

For each of the following periodic signals

- sketch the given function over a few periods

- find the trigonometric Fourier coefficients

- write out the first few terms of the Fourier series.

1. $f(t) = \begin{cases} 1 & 0 < t < \pi/2 \\ 0 & \pi/2 < t < 2\pi \end{cases}$ $\qquad$ $f(t + 2\pi) = f(t)$ $\qquad$ square wave

2. $f(t) = t^2 \quad -1 < t < 1$ $\qquad$ $f(t + 2) = f(t)$

3. $f(t) = \begin{cases} -1 & -T/2 < t < 0 \\ 1 & 0 < t < T/2 \end{cases}$ $\qquad$ $f(t + T) = f(t)$ $\qquad$ square wave

4. $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t^2 & 0 < t < \pi \end{cases}$ $\qquad$ $f(t + 2\pi) = f(t)$

5. $f(t) = \begin{cases} 0 & -T/2 < t < 0 \\ A \sin \dfrac{2\pi t}{T} & 0 < t < T/2 \end{cases}$ $\qquad$ $f(t + T) = f(t)$ $\qquad$ half-wave rectifier

Answers

1.

$$\frac{1}{4} + \frac{1}{\pi}\left\{ \cos t - \frac{\cos 3t}{3} + \frac{\cos 5t}{5} - \ldots \right\}$$

$$+ \frac{1}{\pi}\left\{ \sin t + \frac{2\sin 2t}{2} + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \frac{2\sin 6t}{6} + \ldots \right\}$$

2. $\dfrac{1}{3} - \dfrac{4}{\pi^2}\left\{ \cos \pi t - \dfrac{\cos 2\pi t}{4} + \dfrac{\cos 3\pi t}{9} - \dfrac{\cos 4\pi t}{16} + \ldots \right\}$

3. $\dfrac{4}{\pi}\left\{ \sin \omega t + \dfrac{1}{3}\sin 3\omega t + \dfrac{1}{5}\sin 5\omega t + \ldots \right\}$ $\qquad$ where $\omega = 2\pi/T$.

4.

$$\frac{\pi^2}{6} - 2\left\{ \cos t - \frac{\cos 2t}{2^2} + \frac{\cos 3t}{3^2} - \ldots \right\}$$

$$+ \left\{ \left(\pi - \frac{4}{\pi}\right)\sin t - \frac{\pi}{2}\sin 2t + \left(\frac{\pi}{3} - \frac{4}{3^3\pi}\right)\sin 3t - \frac{\pi}{4}\sin 4t + \ldots \right\}$$

5. $\dfrac{A}{\pi} + \dfrac{A}{2}\sin \omega t - \dfrac{2A}{\pi}\left\{ \dfrac{\cos 2\omega t}{(1)(3)} + \dfrac{\cos 4\omega t}{(3)(5)} + \ldots \right\}$

Even and Odd Functions

Introduction

In this Section we examine how to obtain Fourier series of periodic functions which are either *even* or *odd*. We show that the Fourier series for such functions is considerably easier to obtain as, if the signal is *even* only cosines are involved whereas if the signal is *odd* then only sines are involved. We also show that if a signal reverses after half a period then the Fourier series will only contain odd harmonics.

Prerequisites Before starting this Section you should ...	• know how to obtain a Fourier series • be able to integrate functions involving sinusoids • have knowledge of integration by parts
Learning Outcomes On completion you should be able to ...	• determine if a function is even or odd or neither • easily calculate Fourier coefficients of even or odd functions

1. Even and odd functions

We have shown in the previous Section how to calculate, by integration, the coefficients a_n ($n = 0, 1, 2, 3, \ldots$) and b_n ($n = 1, 2, 3, \ldots$) in a Fourier series. Clearly this is a somewhat tedious process and it is advantageous if we can obtain as much information as possible without recourse to integration. In the previous Section we showed that the square wave (one period of which shown in Figure 12) has a Fourier series containing a constant term and cosine terms only (i.e. all the Fourier coefficients b_n are zero) while the function shown in Figure 13 has a more complicated Fourier series containing both cosine and sine terms as well as a constant.
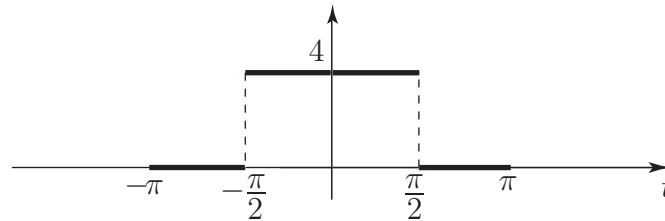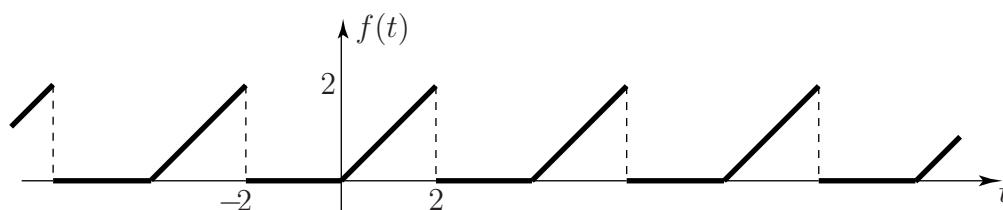
Figure 12: Square wave

Figure 13: Saw-tooth wave

Task

Contrast the symmetry or otherwise of the functions in Figures 12 and 13.

Your solution

Answer
The square wave in Figure 12 has a graph which is symmetrical about the y-axis and is called an **even** function. The saw-tooth wave shown in Figure 13 has no particular symmetry.

In general a function is called **even** if its graph is unchanged under reflection in the y-axis. This is equivalent to

$$f(-t) = f(t) \qquad \text{for all } t$$

Obvious examples of even functions are $t^2, t^4, |t|, \cos t, \cos^2 t, \sin^2 t, \cos nt$.
A function is said to be **odd** if its graph is symmetrical about the origin (i.e. it has rotational symmetry about the origin). This is equivalent to the condition

$$f(-t) = -f(t)$$

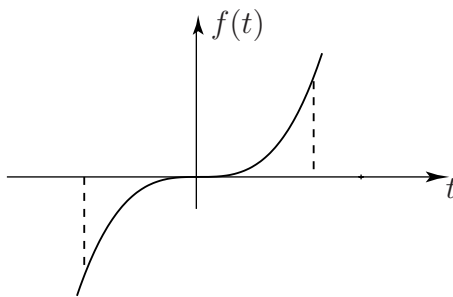Figure 14 shows an example of an odd function.

Figure 14

Examples of odd functions are $t, t^3, \sin t, \sin nt$. A periodic function which is odd is the saw-tooth wave in Figure 15.
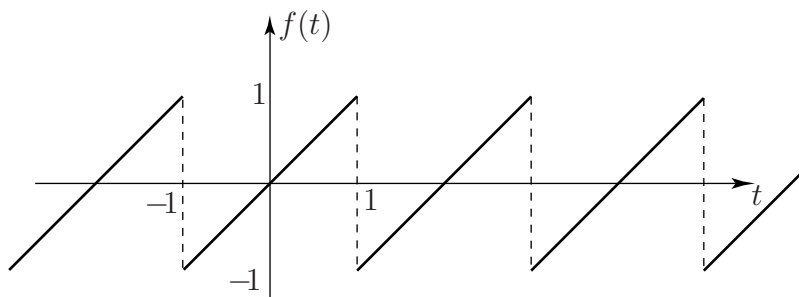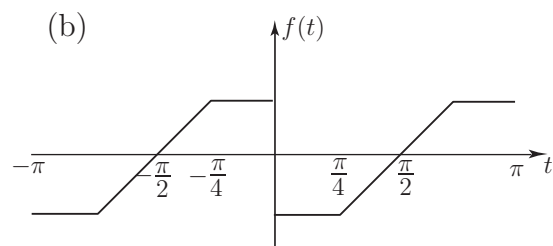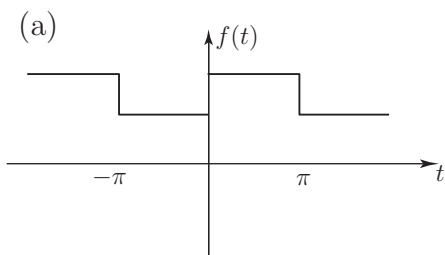

Figure 15

Some functions are **neither even nor odd**. The periodic saw-tooth wave of Figure 13 is an example; another is the exponential function e^t.

Task

State the period of each of the following periodic functions and say whether it is even or odd or neither.

(a)

(b)

Your solution

Answer

(a) is neither even nor odd (with period 2π)

(b) is odd (with period π).

A Fourier series contains a *sum* of terms while the integral formulae for the Fourier coefficients a_n and b_n contain *products* of the type $f(t)\cos nt$ and $f(t)\sin nt$. We need therefore results for sums and products of functions.

Suppose, for example, $g(t)$ is an odd function and $h(t)$ is an even function.

$$
\begin{aligned}
\text{Let} \qquad F_1(t) &= g(t)\,h(t) && \text{(product of odd and even functions)}\\
\text{so} \qquad F_1(-t) &= g(-t)h(-t) && \text{(replacing } t \text{ by } -t)\\
&= (-g(t))h(t) && \text{(since } g \text{ is odd and } h \text{ is even)}\\
&= -g(t)h(t)\\
&= -F_1(t)
\end{aligned}
$$

So $F_1(t)$ is odd.

Now suppose
$$
\begin{aligned}
F_2(t) &= g(t) + h(t) && \text{(sum of odd and even functions)}\\
\therefore \qquad F_2(-t) &= g(-t) + h(t)\\
&= -g(t) + h(t)
\end{aligned}
$$

We see that
$$
\begin{aligned}
F_2(-t) &\neq F_2(t)\\
\text{and} \qquad F_2(-t) &\neq -F_2(t)
\end{aligned}
$$

So $F_2(t)$ is neither even nor odd.

Task

Investigate the odd/even nature of sums and products of
(a) two odd functions $g_1(t), g_2(t)$
(b) two even functions $h_1(t), h_2(t)$

Your solution

Answer

$$\begin{aligned} G_1(t) &= g_1(t)g_2(t) \\ G_1(-t) &= (-g_1(t))(-g_2(t)) \\ &= g_1(t)g_2(t) \\ &= G_1(t) \end{aligned}$$

so the product of two odd functions is even.

$$\begin{aligned} G_2(t) &= g_1(t) + g_2(t) \\ G_2(-t) &= g_1(-t) + g_2(-t) \\ &= -g_1(t) - g_2(t) \\ &= -G_2(t) \end{aligned}$$

so the sum of two odd functions is odd.

$$\begin{aligned} H_1(t) &= h_1(t)h_2(t) \\ H_2(t) &= h_1(t) + h_2(t) \end{aligned}$$

A similar approach shows that

$$\begin{aligned} H_1(-t) &= H_1(t) \\ H_2(-t) &= H_2(t) \end{aligned}$$

i.e. both the sum and product of two even functions are even.

These results are summarized in the following Key Point.

Key Point 5

Products of functions

$$\begin{aligned} (\text{even}) \times (\text{even}) &= (\text{even}) \\ (\text{even}) \times (\text{odd}) &= (\text{odd}) \\ (\text{odd}) \times (\text{odd}) &= (\text{even}) \end{aligned}$$

Sums of functions

$$\begin{aligned} (\text{even}) + (\text{even}) &= (\text{even}) \\ (\text{even}) + (\text{odd}) &= (\text{neither}) \\ (\text{odd}) + (\text{odd}) &= (\text{odd}) \end{aligned}$$

Useful properties of even and of odd functions in connection with integrals can be readily deduced if we recall that a definite integral has the significance of giving us the value of an area:
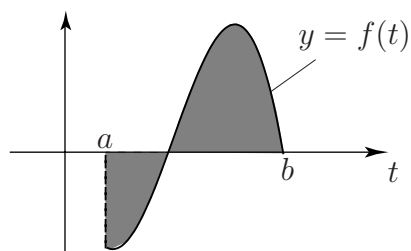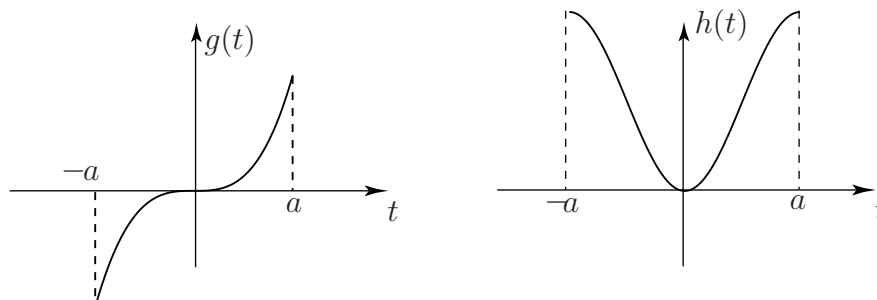

Figure 16

$\int_a^b f(t)\,dt$ gives us the **net** value of the shaded area, that above the t-axis being positive, that below being negative.

Task

For the case of a symmetrical interval $(-a, a)$ deduce what you can about

$$\int_{-a}^a g(t)\,dt \qquad \text{and} \qquad \int_{-a}^a h(t)\,dt$$

where $g(t)$ is an odd function and $h(t)$ is an even function.

Your solution

Answer
We have

$$\int_{-a}^a g(t)\,dt = 0 \qquad \text{for an } \textbf{odd} \text{ function}$$

$$\int_{-a}^a h(t)\,dt = 2\int_0^a h(t)\,dt \quad \text{for an } \textbf{even} \text{ function}$$

(Note that neither result holds for a function which is neither even nor odd.)

2. Fourier series implications

Since a sum of even functions is itself an even function it is not unreasonable to suggest that a Fourier series containing only cosine terms (and perhaps a constant term which can also be considered as an even function) can only represent an even periodic function. Similarly a series of sine terms (and no constant) can only represent an odd function. These results can readily be shown more formally using the expressions for the Fourier coefficients a_n and b_n.

Task

Recall that for a 2π-periodic function

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt$$

If $f(t)$ is even, deduce whether the integrand is even or odd (or neither) and hence evaluate b_n. Repeat for the Fourier coefficients a_n.

Your solution

Answer

We have, if $f(t)$ is even,

$$f(t) \sin nt = (\text{even}) \times (\text{odd}) = \text{odd}$$

hence $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} (\text{odd function})\ dt = 0$

Thus an even function has no sine terms in its Fourier series.

Also $f(t) \cos nt = (\text{even}) \times (\text{even}) = \text{even}$

$$\therefore \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{even function})\, dt = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos nt\, dt.$$

It should be obvious that, for an odd function $f(t)$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{odd function})\, dt = 0$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(t) \sin nt\, dt$$

Analogous results hold for functions of any period, not necessarily 2π.

For a periodic function which is neither even nor odd we can expect at least some of both the a_n and b_n to be non-zero. For example consider the square wave function:
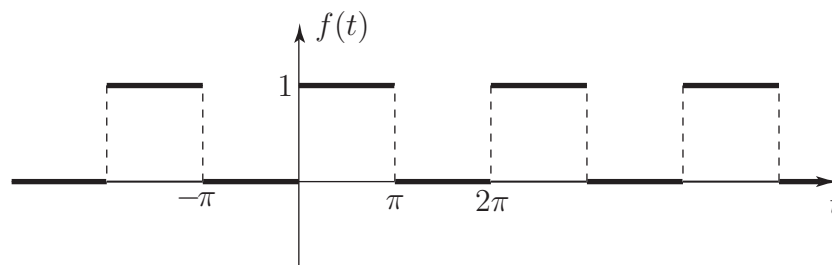

Figure 17: Square wave

This function is neither even nor odd and we have already seen in Section 23.2 that its Fourier series contains a constant $\left(\frac{1}{2}\right)$ and sine terms.

This result could be expected because we can write

$$f(t) = \frac{1}{2} + g(t)$$

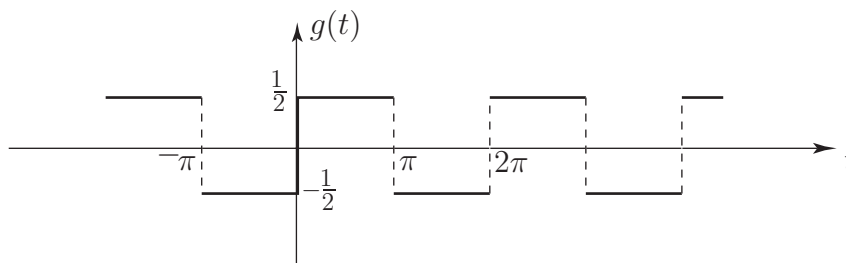where $g(t)$ is as shown:

Figure 18

Clearly $g(t)$ is odd and will contain only sine terms. The Fourier series are in fact

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \dots\right)$$

and

$$g(t) = \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \dots\right)$$

Task

For each of the following functions deduce whether the corresponding Fourier series contains

(a) sine terms only or cosine terms only or both

(b) a constant term
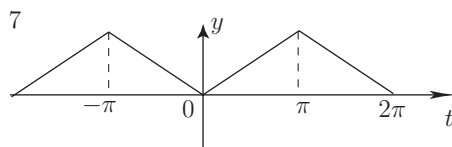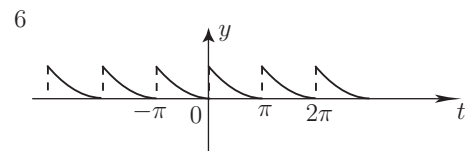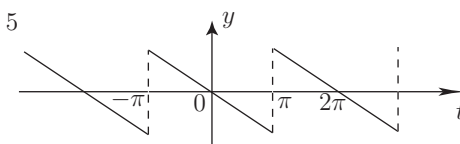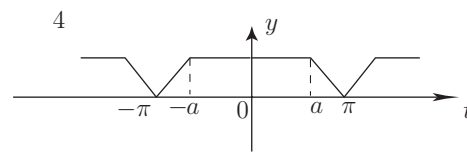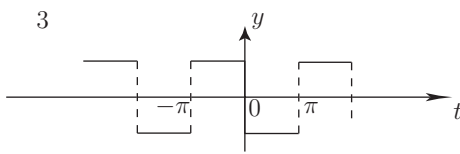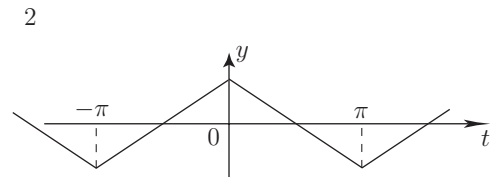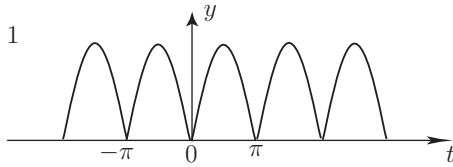
Your solution

Answer

1. cosine terms only (plus constant).

2. cosine terms only (no constant).

3. sine terms only (no constant).

4. cosine terms only (plus constant).

5. sine terms only (no constant).

6. sine and cosine terms (plus constant).

7. cosine terms only (plus constant).

Task

Confirm the result obtained for the triangular wave, function 7 in the last Task, by finding the Fourier series fully. The function involved is

$$f(t) = |t| \qquad -\pi < t < \pi$$
$$f(t + 2\pi) = f(t)$$

Your solution

Answer

Since $f(t)$ is even we can say immediately

$$b_n = 0 \qquad n = 1, 2, 3, \ldots$$

Also

$$a_n = \frac{2}{\pi} \int_0^\pi t \cos nt \, dt = \begin{cases} 0 & n \text{ even} \\[2ex] -\dfrac{4}{n^2 \pi} & n \text{ odd} \end{cases} \qquad \text{(after integration by parts)}$$

Also $a_0 = \dfrac{2}{\pi} \int_0^\pi t \, dt = \pi$ so the Fourier series is

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos t + \frac{1}{9} \cos 3t + \frac{1}{25} \cos 5t + \ldots \right)$$

Convergence

Introduction

In this Section we examine, briefly, the convergence characteristics of a Fourier series. We have seen that a Fourier series can be found for functions which are not necessarily continuous (there may be *jumps* in the curve) − the only requirement that we have made is that the function be periodic. We have seen that the more terms we take in the Fourier series the better is the approximation to the given signal. But an obvious question to ask is *what happens at the points of discontinuity?* What does the Fourier series converge to at these points? It must converge to something (finite) since a Fourier series is a sum of very smooth continuous functions. In this Section we give the answer to this question.

Prerequisites

Before starting this Section you should . . .

- know how to obtain a Fourier series
- be familiar with the limit process as applied to functions

Learning Outcomes

On completion you should be able to . . .

- determine what a Fourier series converges to at each point, including at a point of discontinuity
- use the convergence property of Fourier Series to obtain series for the number π

1. Convergence of a Fourier series

We have now shown how to obtain a Fourier series for periodic functions. We have suggested that we would expect to be able to approximate such functions by using a few terms of the Fourier series. The detailed question of the **convergence** or otherwise of Fourier series has not been discussed. The reason for this is that the great majority of functions likely to be encountered in practice have Fourier series that do indeed converge and can therefore be safely used as approximations.

The precise conditions that have to be fulfilled for a Fourier series to converge are known as Dirichlet conditions after the French mathematician who investigated the matter. The three conditions are listed in the following Key Point.

Key Point 6

The Dirichlet conditions for the convergence of a Fourier series of a periodic function $f(t)$ are:

1. $f(t)$ must have only a finite number of finite discontinuities, within one period

2. $f(t)$ must have a finite number of maxima and minima over one period

3. the integral $\displaystyle\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|\, dt$ must be finite.

It follows, for example, that if $f(t)$ is defined over $(-\pi, \pi)$ as one of the following functions t^3 or $1/(t-4)$ or $3t+2$ and $f(t+2\pi) = f(t)$, then $f(t)$ can indeed be represented as a Fourier series as each function satisfies the Dirichlet conditions.

On the other hand, if, over $(-\pi, \pi)$, $f(t)$ is $\dfrac{1}{t}$ or $\dfrac{1}{t-2}$ or $\tan t$ then $f(t)$ cannot be expanded in a Fourier series because each of these functions has an infinite discontinuity within $(-\pi, \pi)$.

If the Dirichlet conditions are satisfied at a point $t = t_0$ where $f(t)$ is continuous then, as we would expect, the Fourier series at t_0 given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{ a_n \cos\left(\frac{2n\pi t_0}{T}\right) + b_n \sin\left(\frac{2n\pi t_0}{T}\right) \right\} \qquad \text{converges to the function value } f(t_0)$$

At a point, say $t = t_1$, at which $f(t)$ has a discontinuity then the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{ a_n \cos\left(\frac{2n\pi t_1}{T}\right) + b_n \sin\left(\frac{2n\pi t_1}{T}\right) \right\} \qquad \text{converges to} \quad \frac{1}{2}\left\{ f(t_{1-}) + f(t_{1+}) \right\}$$

where $f(t_{1-})$ is the limit of $f(t)$ as t approaches t_1 from the left and $f(t_{1+})$ is the limit as t approaches t_1 from the right (Figure 19).
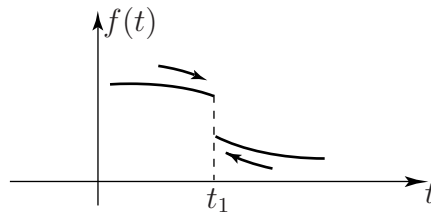
Figure 19

Key Point 7

If Dirichlet conditions are satisfied then at a point of continuity $t = t_o$

$$f(t_0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t_0}{T}\right) + b_n \sin\left(\frac{2n\pi t_0}{T}\right) \right\}$$

whereas at a point of discontinuity $t = t_1$ the Fourier series converges to the **average** of the two limiting values

$$\frac{1}{2}\left\{f(t_{1-}) + f(t_{1+})\right\} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t_1}{T}\right) + b_n \sin\left(\frac{2n\pi t_1}{T}\right) \right\}$$

Example 2
Suppose we consider the square wave
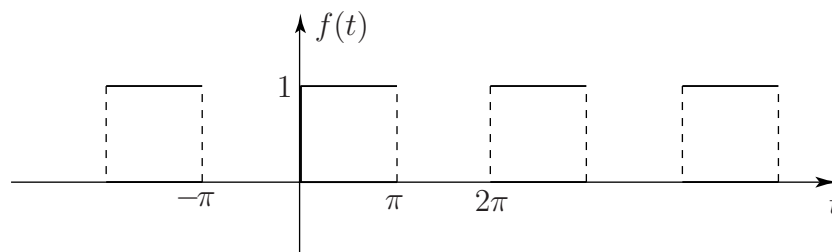
Figure 20

Here $f(t)$ has finite discontinuities at $-\pi, 0$ and π. The Fourier series of $f(t)$ is (see Section 23.3, subsection 2)

$$\frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \dots\right).$$

At $t = \dfrac{\pi}{2}$, for example, where $f(t)$ is continuous the square wave converges to $f\left(\dfrac{\pi}{2}\right) = 1$. On the other hand at $t = \pi$ the Fourier series clearly has the value $\dfrac{1}{2}$ (since all the sine terms are zero here). This value $\dfrac{1}{2}$ agrees with the average of the two limiting values of $f(t)$ at $t = \dfrac{\pi}{2}$: $\dfrac{1}{2}(1 + 0) = \dfrac{1}{2}$. If we actually put $t = \dfrac{\pi}{2}$ in the Fourier series we obtain

$$\dfrac{1}{2} + \dfrac{2}{\pi}\left(\sin\left(\dfrac{\pi}{2}\right) + \dfrac{1}{3}\sin\left(\dfrac{3\pi}{2}\right) + \dfrac{1}{5}\sin\left(\dfrac{5\pi}{2}\right) + \ldots\right)$$

$$= \dfrac{1}{2} + \dfrac{2}{\pi}\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots\right)$$

Since the series converges, as we have seen, to $f\left(\dfrac{\pi}{2}\right) = 1$, we obtain the interesting result

$$\dfrac{1}{2} + \dfrac{2}{\pi}\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots\right) = 1 \quad \text{or} \quad 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots \quad = \dfrac{\pi}{4}$$

Task

The function
$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t^2 & 0 < t < \pi \end{cases}$$

$$f(t + 2\pi) = f(t)$$
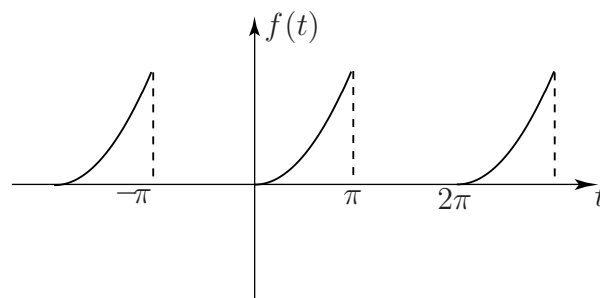
has Fourier series (see Exercise 4 at the end of Section 23.2)

$$\dfrac{\pi^2}{6} - 2\left(\cos t - \dfrac{\cos 2t}{4} + \dfrac{\cos 3t}{9} - \ldots\right)$$

$$+ \left\{\left(\pi - \dfrac{4}{\pi}\right)\sin t - \dfrac{\pi}{2}\sin 2t + \left(\dfrac{\pi}{3} - \dfrac{4}{9\pi}\right)\sin 3t - \dfrac{\pi}{4}\sin 4t + \ldots\right\}$$

By using a suitable value of t show that

$$1 + \dfrac{1}{4} + \dfrac{1}{9} + \dfrac{1}{16} + \ldots = \dfrac{\pi^2}{6}$$

First decide on the appropriate value of t to use:

Your solution

Answer
Looking at the Fourier series, the numerical series we seek is present in the cosine terms so we need to remove the sine terms. This we can do by selecting $t = 0$ or $t = \pi$. The choice $t = 0$ will make the cosine terms become:

$$1 - \frac{1}{4} + \frac{1}{9} - \dots$$

which is not what we seek. Hence we put $t = \pi$.

Now put $t = \pi$ in the series and decide what the Fourier series will converge to at this value. Hence complete the question:

Your solution

Answer
At $t = \pi$ the Fourier series is

$$\frac{\pi^2}{6} - 2\left(\cos \pi - \frac{\cos 2\pi}{4} + \frac{\cos 3\pi}{9} - \dots\right)$$

$$= \frac{\pi^2}{6} - 2\left(-1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \dots\right) = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right)$$

At $t = \pi$ the Fourier series will converge to

$$\frac{1}{2}\left(\pi^2 + 0\right) = \frac{\pi^2}{2} \quad \text{(the average of the left and right hand limits)}$$

So $\quad \frac{\pi^2}{6} + 2\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots\right) = \frac{\pi^2}{2} \qquad \therefore \qquad 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = \frac{\pi^2}{6}$

Note that in the last Task if we substitute $t = 0$ in the Fourier series (which converges to $f(0) = 0$) we obtain another infinite series but with alternating signs:

$$\frac{\pi^2}{6} - 2\left(1 - \frac{1}{4} + \frac{1}{9} - \dots\right) = 0 \quad \text{or} \quad 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots \qquad = \frac{\pi^2}{12}$$

Exercises

1. Obtain the Fourier series of

$$f(t) = |t| \qquad -\pi \le t \le \pi \qquad f(t + 2\pi) = f(t)$$

By putting $t = 0$ show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

2. (a) Obtain the Fourier series of the 2π periodic function

$$f(t) = \frac{t^2}{4} \qquad -\pi \le t \le \pi$$

and use it to obtain the following identities:

(i) $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{6}$ (ii) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{12}$

(b) Show that $1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \dfrac{1}{7^2} \cdots = \dfrac{\pi^2}{8}$

3. Obtain the Fourier series of the 2π periodic function

$$f(t) = t \qquad -\pi \le t \le \pi$$

Use the series to show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Answers

1. $\dfrac{\pi}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-4)}{(2n-1)^2 \pi} \cos[(2n-1)t]$

2. (a) $\dfrac{\pi^2}{12} + \displaystyle\sum_{n=1}^{\infty} \dfrac{\cos(n\pi)}{n^2} \cos nt$ (i) Put $t = \pi$ (ii) Put $t = 0$

   (b) Add the two series from (a).

3. $-2 \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n} \sin nt$

Half-Range Series

Introduction

In this Section we address the following problem:

> *Can we find a Fourier series expansion of a function defined over a finite interval?*

Of course we recognise that such a function could not be periodic (as periodicity demands an infinite interval). The answer to this question is yes but we must first convert the given non-periodic function into a periodic function. There are many ways of doing this. We shall concentrate on the most useful extension to produce a so-called **half-range Fourier series**.

Prerequisites Before starting this Section you should ...	• know how to obtain a Fourier series • be familiar with odd and even functions and their properties • have knowledge of integration by parts
Learning Outcomes On completion you should be able to ...	• choose to expand a non-periodic function either as a series of sines or as a series of cosines

1. Half-range Fourier series

So far we have shown how to represent given periodic functions by Fourier series. We now consider a slight variation on this theme which will be useful in HELM 25 on solving Partial Differential Equations.

Suppose that instead of specifying a periodic function we begin with a function $f(t)$ defined only over a **limited range of values** of t, say $0 < t < \pi$. Suppose further that we wish to represent this function, over $0 < t < \pi$, by a Fourier series. (This situation may seem a little artificial at this point, but this is precisely the situation that will arise in solving differential equations.)

To be specific, suppose we define $f(t) = t^2 \qquad 0 < t < \pi$
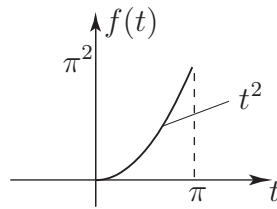
Figure 21

We shall consider the interval $0 < t < \pi$ to be half a period of a 2π periodic function. We must therefore define $f(t)$ for $-\pi < t < 0$ to complete the specification.

Task

Complete the definition of the above function $f(t) = t^2, \; 0 < t < \pi$
by defining it over $-\pi < t < 0$ such that the resulting functions will have a Fourier series containing

(a) only cosine terms, (b) only sine terms, (c) both cosine and sine terms.

Your solution

Answer
(a) We must complete the definition so as to have an **even** periodic function:
$f(t) = t^2, \quad -\pi < t < 0$
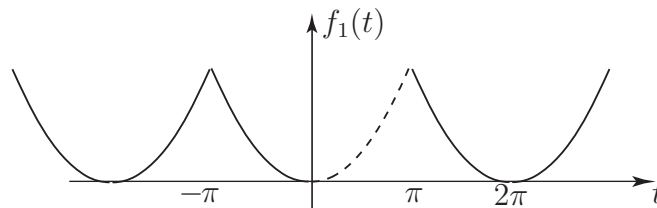
(b) We must complete the definition so as to have an **odd** periodic function:
$f(t) = -t^2, \quad -\pi < t < 0$
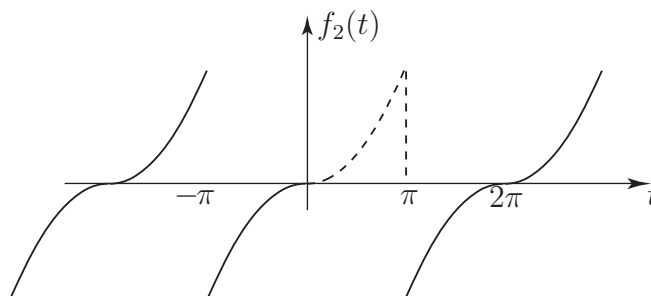
(c) We may define $f(t)$ in any way we please (other than (a) and (b) above). For example we might define $f(t) = 0$ over $-\pi < t < 0$:
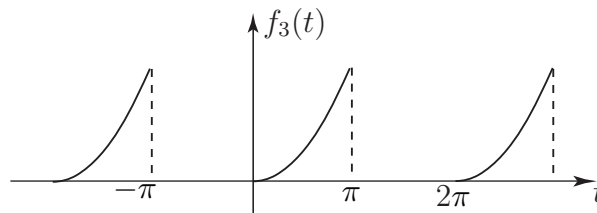
The point is that all three periodic functions $f_1(t), f_2(t), f_3(t)$ will give rise to a **different** Fourier series but all will represent the function $f(t) = t^2$ over $0 < t < \pi$. Fourier series obtained by extending functions in this sort of way are often referred to as **half-range** series.

Normally, in applications, we require either a Fourier Cosine series (so we would complete a definition as in (i) above to obtain an **even** periodic function) or a Fourier Sine series (for which, as in (ii) above, we need an **odd** periodic function.)

The above considerations apply equally well for a function defined over any interval.

Example 3

Obtain the half range Fourier Sine series to represent $f(t) = t^2 \qquad 0 < t < 3$.

Solution

We first extend $f(t)$ as an odd periodic function $F(t)$ of **period 6**: $f(t) = -t^2, \quad -3 < t < 0$
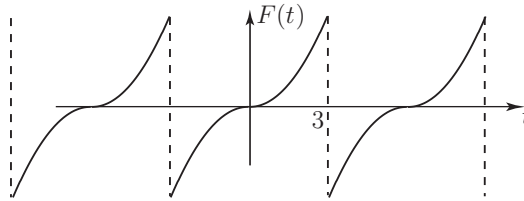
Figure 22

We now evaluate the Fourier series of $F(t)$ by standard techniques but take advantage of the symmetry and put $a_n = 0, \ n = 0, 1, 2, \ldots$.

Using the results for the Fourier Sine coefficients for period T from HELM 23.2 subsection 5,

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} F(t) \sin\left(\frac{2n\pi t}{T}\right) dt,$$

we put $T = 6$ and, since the integrand is even (a product of 2 odd functions), we can write

$$b_n = \frac{2}{3} \int_0^3 F(t) \sin\left(\frac{2n\pi t}{6}\right) dt = \frac{2}{3} \int_0^3 t^2 \sin\left(\frac{n\pi t}{3}\right) dt.$$

(Note that we always integrate over the originally defined range, in this case $0 < t < 3$.)

We now have to integrate by parts (twice!)

$$b_n = \frac{2}{3} \left\{ \left[-\frac{3t^2}{n\pi} \cos\left(\frac{n\pi t}{3}\right) \right]_0^3 + 2\left(\frac{3}{n\pi}\right) \int_0^3 t \cos\left(\frac{n\pi t}{3}\right) dt \right\}$$

$$= \frac{2}{3} \left\{ -\frac{27}{n\pi} \cos n\pi + \frac{6}{n\pi} \left[\frac{3}{n\pi} t \sin \frac{n\pi t}{3} \right]_0^3 - \left(\frac{6}{n\pi}\right)\left(\frac{3}{n\pi}\right) \int_0^3 \sin\left(\frac{n\pi t}{3}\right) dt \right\}$$

$$= \frac{2}{3} \left\{ -\frac{27}{n\pi} \cos n\pi - \frac{18}{n^2\pi^2} \left[-\frac{3}{n\pi} \cos\left(\frac{n\pi t}{3}\right) \right]_0^3 \right\} = \frac{2}{3} \left\{ -\frac{27}{n\pi} \cos n\pi + \frac{54}{n^3\pi^3} (\cos n\pi - 1) \right\}$$

$$= \begin{cases} -\dfrac{18}{n\pi} & n = 2, 4, 6, \ldots \\[3mm] \dfrac{18}{n\pi} - \dfrac{72}{n^3\pi^3} & n = 1, 3, 5, \ldots \end{cases}$$

So the required Fourier Sine series is

$$F(t) = 18\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) \sin\left(\frac{\pi t}{3}\right) - \frac{18}{2\pi} \sin\left(\frac{2\pi t}{3}\right) + 18\left(\frac{1}{3\pi} - \frac{4}{27\pi^3}\right) \sin(\pi t) - \ldots$$

Task

Obtain a half-range Fourier Cosine series to represent the function

$$f(t) = 4 - t \qquad 0 < t < 4.$$
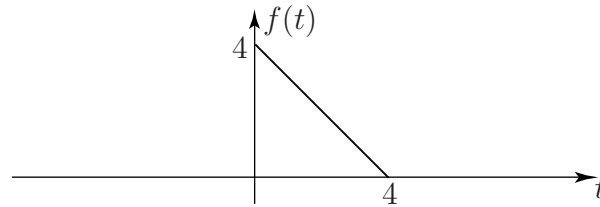
First complete the definition to obtain an even periodic function $F(t)$ of period 8. Sketch $F(t)$:

Your solution

Answer
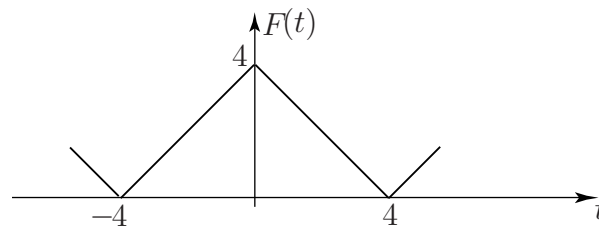
Now formulate the integral from which the Fourier coefficients a_n can be calculated:

Your solution

Answer
We have with $T = 8$

$$a_n = \frac{2}{8} \int_{-4}^{4} F(t) \cos\left(\frac{2n\pi t}{8}\right) dt$$

Utilising the fact that the integrand here is even we get

$$a_n = \frac{1}{2} \int_{0}^{4} (4 - t) \cos\left(\frac{n\pi t}{4}\right) dt$$

Now integrate by parts to obtain a_n and also obtain a_0:

Your solution

Answer

Using integration by parts we obtain for $n = 1, 2, 3, \ldots$

$$a_n = \frac{1}{2} \left\{ \left[(4 - t) \frac{4}{n\pi} \sin \left(\frac{n\pi t}{4} \right) \right]_0^4 + \frac{4}{n\pi} \int_0^4 \sin \left(\frac{n\pi t}{4} \right) dt \right\}$$

$$= \frac{1}{2} \left(\frac{4}{n\pi} \right) \left(\frac{4}{n\pi} \right) \left[-\cos \left(\frac{n\pi t}{4} \right) \right]_0^4 = \frac{8}{n^2 \pi^2} \left[-\cos(n\pi) + 1 \right]$$

i.e. $\quad a_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\[2mm] \dfrac{16}{n^2 \pi^2} & n = 1, 3, 5, \ldots \end{cases}$

Also $a_0 = \dfrac{1}{2} \displaystyle\int_0^4 (4 - t)\, dt = 4$. So the constant term is $\dfrac{a_0}{2} = 2$.

Now write down the required Fourier series:

Your solution

Answer

We get $\quad 2 + \dfrac{16}{\pi^2} \left\{ \cos \left(\dfrac{\pi t}{4} \right) + \dfrac{1}{9} \cos \left(\dfrac{3\pi t}{4} \right) + \dfrac{1}{25} \cos \left(\dfrac{5\pi t}{4} \right) + \ldots \right\}$

Note that the form of the Fourier series (a constant of 2 together with odd harmonic cosine terms) could be predicted if, in the sketch of $F(t)$, we imagine raising the t-axis by 2 units i.e. writing

$$F(t) = 2 + G(t)$$
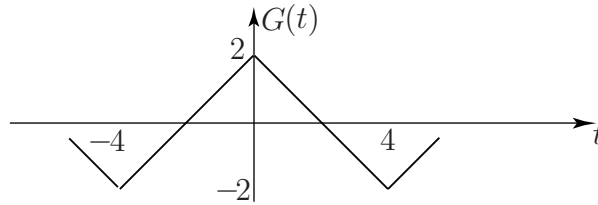
Figure 23

Clearly $G(t)$ possesses half-period symmetry

$$G(t + 4) = -G(t)$$

and hence its Fourier series must contain only odd harmonics.

Exercises

Obtain the half-range Fourier series specified for each of the following functions:

1. $f(t) = 1 \qquad 0 \le t \le \pi$ (sine series)

2. $f(t) = t \qquad 0 \le t \le 1$ (sine series)

3. (a) $f(t) = e^{2t} \qquad 0 \le t \le 1$ (cosine series)
   (b) $f(t) = e^{2t} \qquad 0 \le t \le \pi$ (sine series)

4. (a) $f(t) = \sin t \qquad 0 \le t \le \pi$ (cosine series)
   (b) $f(t) = \sin t \qquad 0 \le t \le \pi$ (sine series)

Answers

1. $\dfrac{4}{\pi} \left\{ \sin t + \dfrac{1}{3} \sin 3t + \dfrac{1}{5} \sin 5t + \cdots \right\}$

2. $\dfrac{2}{\pi} \{ \sin \pi t - \dfrac{1}{2} \sin 2\pi t + \dfrac{1}{3} \sin 3\pi t - \cdots \}$

3. (a) $\dfrac{e^2 - 1}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{4 + n^2 \pi^2} \{ e^2 \cos(n\pi) - 1 \} \cos n\pi t$

   (b) $\displaystyle\sum_{n=1}^{\infty} \dfrac{2n\pi}{4 + n^2 \pi^2} \{ 1 - e^2 \cos(n\pi) \} \sin n\pi t$

4. (a) $\dfrac{2}{\pi} + \displaystyle\sum_{n=2}^{\infty} \dfrac{1}{\pi} \left\{ \dfrac{1}{1-n}(1 - \cos(1-n)\pi) + \dfrac{1}{1+n}(1 - \cos(1+n)\pi) \right\} \cos nt$

   (b) $\sin t$ itself (!)

The Complex Form

➤ Introduction

In this Section we show how a Fourier series can be expressed more concisely if we introduce the complex number i where $i^2 = -1$. By utilising the Euler relation:

$$e^{i\theta} \equiv \cos\theta + i\sin\theta$$

we can replace the trigonometric functions by complex exponential functions. By also combining the Fourier coefficients a_n and b_n into a complex coefficient c_n through

$$c_n = \frac{1}{2}(a_n - ib_n)$$

we find that, for a given periodic signal, both sets of constants can be found in one operation.

We also obtain Parseval's theorem which has important applications in electrical engineering.

The complex formulation of a Fourier series is an important precursor of the Fourier transform which attempts to Fourier analyse non-periodic functions.

➤ Prerequisites

Before starting this Section you should . . .

- know how to obtain a Fourier series

- be competent working with the complex numbers

- be familiar with the relation between the exponential function and the trigonometric functions

➤ Learning Outcomes

On completion you should be able to . . .

- express a periodic function in terms of its Fourier series in complex form

- understand Parseval's theorem

1. Complex exponential form of a Fourier series

So far we have discussed the **trigonometric** form of a Fourier series i.e. we have represented functions of period T in the terms of sinusoids, and possibly a constant term, using

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right\}.$$

If we use the angular frequency

$$\omega_0 = \frac{2\pi}{T}$$

we obtain the more concise form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t).$$

We have seen that the Fourier coefficients are calculated using the following integrals:

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\omega_0 t \; dt \qquad n = 0, 1, 2, \dots \tag{1}$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin n\omega_0 t \; dt \qquad n = 1, 2, \dots \tag{2}$$

An alternative, more concise form, of a Fourier series is available using **complex quantities**. This form is quite widely used by engineers, for example in Circuit Theory and Control Theory, and leads naturally into the Fourier Transform which is the subject of HELM 24.

2. Revision of the exponential form of a complex number

Recall that a complex number in **Cartesian** form which is written as

$$z = a + \mathrm{i}b,$$

where a and b are real numbers and $\mathrm{i}^2 = -1$, can be written in **polar** form as

$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

where $r = |z| = \sqrt{a^2 + b^2}$ and θ, the **argument** or **phase** of z, is such that

$$a = r\cos\theta \qquad b = r\sin\theta.$$

A more concise version of the polar form of z can be obtained by defining a **complex exponential** quantity $e^{\mathrm{i}\theta}$ by Euler's relation

$$e^{\mathrm{i}\theta} \equiv \cos\theta + \mathrm{i}\sin\theta$$

The polar angle θ is normally expressed in **radians**. Replacing i by $-\mathrm{i}$ we obtain the alternative form

$$e^{-\mathrm{i}\theta} \equiv \cos\theta - \mathrm{i}\sin\theta$$

Task

Write down in $\cos\theta \pm i\sin\theta$ form and also in Cartesian form (a) $e^{i\pi/6}$ (b) $e^{-i\pi/6}$.

Use Euler's relation:

Your solution

Answer

We have, by definition,

(a) $e^{i\pi/6} = \cos\left(\dfrac{\pi}{6}\right) + i\sin\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\,i$ (b) $e^{-i\pi/6} = \cos\left(\dfrac{\pi}{6}\right) - i\sin\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\,i$

Task

Write down (a) $\cos\left(\dfrac{\pi}{6}\right)$ (b) $\sin\left(\dfrac{\pi}{6}\right)$ in terms of $e^{i\pi/6}$ and $e^{-i\pi/6}$.

Your solution

Answer

We have, adding the two results from the previous task

$$e^{i\pi/6} + e^{-i\pi/6} = 2\cos\left(\frac{\pi}{6}\right) \qquad \text{or} \qquad \cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\left(e^{i\pi/6} + e^{-i\pi/6}\right)$$

Similarly, subtracting the two results,

$$e^{i\pi/6} - e^{-i\pi/6} = 2\,i\sin\left(\frac{\pi}{6}\right) \qquad \text{or} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2\,i}\left(e^{i\pi/6} - e^{-i\pi/6}\right)$$

(Don't forget the factor i in this latter case.)

Clearly, similar calculations could be carried out for any angle θ. The general results are summarised in the following Key Point.

Euler's Relations

$$\mathrm{e}^{\mathrm{i}\theta} \equiv \cos\theta + \mathrm{i}\sin\theta, \qquad \mathrm{e}^{-\mathrm{i}\theta} \equiv \cos\theta - \mathrm{i}\sin\theta$$

$$\cos\theta \equiv \frac{1}{2}\left(e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta}\right) \qquad \sin\theta \equiv \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\right)$$

Using these results we can redraft an expression of the form

$$a_n \cos n\theta + b_n \sin n\theta$$

in terms of complex exponentials.
(This expression, with $\theta = \omega_0 t$, is of course the nth harmonic of a trigonometric Fourier series.)

Task

Using the results from the Key Point 8 (with $n\theta$ instead of θ) rewrite

$$a_n \cos n\theta + b_n \sin n\theta$$

in complex exponential form.

First substitute for $\cos n\theta$ and $\sin n\theta$ with exponential expressions using Key Point 8:

Your solution

Answer
We have

$$a_n \cos n\theta = \frac{a_n}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right) \qquad b_n \sin n\theta = \frac{b_n}{2\mathrm{i}}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right)$$

so

$$a_n \cos n\theta + b_n \sin n\theta = \frac{a_n}{2}\left(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta}\right) + \frac{b_n}{2\mathrm{i}}\left(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}\right)$$

Now collect the terms in $e^{in\theta}$ and in $e^{-in\theta}$ and use the fact that $\dfrac{1}{i} = -i$:

Your solution

Answer
We get

$$\frac{1}{2}\left(a_n + \frac{b_n}{i}\right) e^{in\theta} + \frac{1}{2}\left(a_n - \frac{b_n}{i}\right) e^{-in\theta}$$

or, since $\dfrac{1}{i} = \dfrac{i}{i^2} = -i$ $\quad \dfrac{1}{2}(a_n - ib_n)e^{in\theta} + \dfrac{1}{2}(a_n + ib_n)e^{-in\theta}$.

Now write this expression in more concise form by defining

$$c_n = \frac{1}{2}(a_n - ib_n) \text{ which has complex conjugate } c_n^* = \frac{1}{2}(a_n + ib_n).$$

Write the concise complex exponential expression for $a_n \cos n\theta + b_n \sin n\theta$:

Your solution

Answer
$$a_n \cos n\theta + b_n \sin n\theta = c_n e^{in\theta} + c_n^* e^{-in\theta}$$

Clearly, we can now rewrite the trigonometric Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{as} \quad \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n e^{in\omega_0 t} + c_n^* e^{-in\omega_0 t}\right) \tag{3}$$

A neater, and particularly concise, form of this expression can be obtained as follows:
Firstly write $\dfrac{a_0}{2} = c_0$ (which is consistent with the general definition of c_n since $b_0 = 0$).
The second term in the summation

$$\sum_{n=1}^{\infty} c_n^* e^{-in\omega_0 t} = c_1^* e^{-i\omega_0 t} + c_2^* e^{-2i\omega_0 t} + \ldots$$

can be written, if we define $c_{-n} = c_n^* = \frac{1}{2}(a_n + ib_n)$, as

$$c_{-1}e^{-i\omega_0 t} + c_{-2}e^{-2i\omega_0 t} + c_{-3}e^{-3i\omega_0 t} + \ldots = \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 t}$$

Hence (3) can be written $\quad c_0 + \displaystyle\sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 t}$ or in the very concise form $\displaystyle\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$.

The **complex Fourier coefficients** c_n can be readily obtained as follows using (1) and (2) for a_n, b_n. Firstly

$$c_0 = \frac{a_0}{2} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, dt \tag{4}$$

For $n = 1, 2, 3, \ldots$ we have

$$c_n = \frac{1}{2}(a_n - \mathrm{i}b_n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)(\cos n\omega_0 t - \mathrm{i}\sin n\omega_0 t)\, dt \quad \text{i.e.} \quad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\,\mathrm{i}n\omega_0 t}\, dt \tag{5}$$

Also for $n = 1, 2, 3, \ldots$ we have

$$c_{-n} = c_n^* = \frac{1}{2}(a_n + \mathrm{i}b_n) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{\mathrm{i}n\omega_0 t}\, dt$$

This last expression is equivalent to stating that for $n = -1, -2, -3, \ldots$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\,\mathrm{i}n\omega_0 t}\, dt \tag{6}$$

The three equations (4), (5), (6) can thus all be contained in the one expression

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\,\mathrm{i}n\omega_0 t}\, dt \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The results of this discussion are summarised in the following Key Point.

Key Point 9

Fourier Series in Complex Form

A function $f(t)$ of period T has a complex Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}n\omega_0 t} \qquad \text{where} \qquad c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-\,\mathrm{i}n\omega_0 t}\, dt$$

For the special case $T = 2\pi$, so that $\omega_0 = 1$, these formulae become particularly simple:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{i}nt} \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-\,\mathrm{i}nt}\, dt.$$

---

3. Properties of the complex Fourier coefficients

Using properties of the trigonometric Fourier coefficients a_n, b_n we can readily deduce the following results for the c_n coefficients:

1. $c_0 = \dfrac{a_0}{2}$ is always real.

2. Suppose the periodic function $f(t)$ is even so that all b_n are zero. Then, since in the complex form the b_n arise as the imaginary part of c_n, it follows that for $f(t)$ even the coefficients c_n $(n = \pm 1, \pm 2, \ldots)$ are wholly real.

Task

If $f(t)$ is odd, what can you deduce about the Fourier coefficients c_n?

Your solution

Answer
Since, for an odd periodic function the Fourier coefficients a_n (which constitute the real part of c_n) are zero, then in this case the complex coefficients c_n are wholly imaginary.

3. Since

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t} dt$$

then if $f(t)$ is even, c_n will be real, and we have two possible methods for evaluating c_n:

(a) Evaluate the integral above **as it stands** i.e. over the full range $\left(-\dfrac{T}{2}, \dfrac{T}{2}\right)$. Note carefully that the second term in the integrand is neither an even nor an odd function so the integrand itself is

(even function) $\times$ (neither even nor odd function) $=$ neither even nor odd function.

Thus we **cannot** write $\quad c_n = \dfrac{2}{T} \int_0^{T/2} f(t) e^{-in\omega_0 t} \, dt$

(b) Put $e^{-in\omega_0 t} = \cos n\omega_0 t - i \sin n\omega_0 t$ so
$$f(t) e^{-in\omega_0 t} = f(t) \cos n\omega_0 t - i f(t) \sin n\omega_0 t \quad = \quad (\text{even})(\text{even}) - i(\text{even})(\text{odd})$$
$$= \quad (\text{even}) - i(\text{odd}).$$

Hence $\quad c_n = \dfrac{2}{T} \int_0^{\frac{T}{2}} f(t) \cos n\omega_0 t \, dt = \dfrac{a_n}{2}.$

4. If $f(t + \dfrac{T}{2}) = -f(t)$ then of course only **odd** harmonic coefficients c_n $(n = \pm 1, \pm 3, \pm 5, \ldots)$ will arise in the complex Fourier series just as with trigonometric series.

Example 4

Find the complex Fourier series of the saw-tooth wave shown in Figure 24:
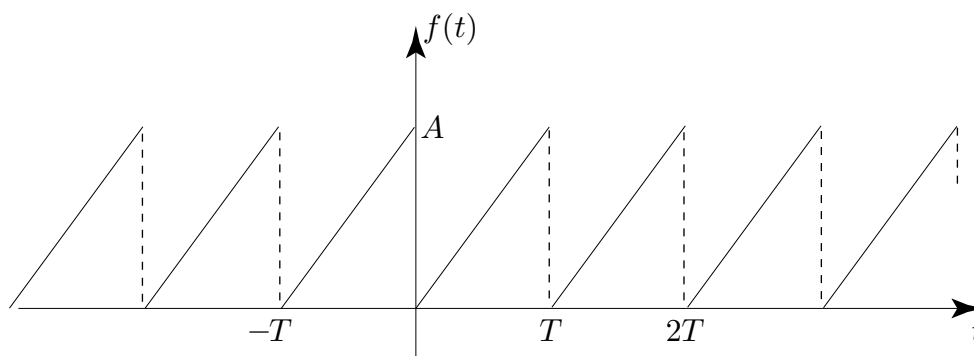

Figure 24

Solution

We have

$$f(t) = \frac{At}{T} \qquad 0 < t < T \qquad f(t+T) = f(t)$$

The period is T in this case so $\omega_0 = \dfrac{2\pi}{T}$.

Looking at the graph of $f(t)$ we can say immediately

    (a) the Fourier series will contain a constant term c_0

    (b) if we imagine shifting the horizontal axis up to $\dfrac{A}{2}$ the signal can be written

        $f(t) = \dfrac{A}{2} + g(t)$, where $g(t)$ is an odd function with complex Fourier coefficients that are purely imaginary.

Hence we expect the required complex Fourier series of $f(t)$ to contain a constant term $\dfrac{A}{2}$ and complex exponential terms with purely imaginary coefficients. We have, from the general theory, and using $0 < t < T$ as the basic period for integrating,

$$c_n = \frac{1}{T} \int_0^T \frac{At}{T} e^{-\,in\omega_0 t}\, dt = \frac{A}{T^2} \int_0^T t e^{-\,in\omega_0 t}\, dt$$

We can evaluate the integral using parts:

$$
\begin{aligned}
\int_0^T t e^{-\,in\omega_0 t}\, dt &= \left[\frac{t e^{-\,in\omega_0 t}}{(-\,in\omega_0)} \right]_0^T + \frac{1}{in\omega_0} \int_0^T e^{-\,in\omega_0 t}\, dt \\[2mm]
&= \frac{T e^{\,in\omega_0 T}}{(-\,in\omega_0)} - \frac{1}{(\,in\omega_0)^2} \left[e^{-\,in\omega_0 t} \right]_0^T
\end{aligned}
$$

Solution (contd.)

But $\omega_0 = \dfrac{2\pi}{T}$ so

$$
\begin{aligned}
e^{-\,in\omega_0 T} = e^{-\,in2\pi} &= \cos 2n\pi - \text{i} \sin 2n\pi \\
&= 1 - 0\,\text{i} = 1
\end{aligned}
$$

Hence the integral becomes

$$
\frac{T}{-\,in\omega_0} - \frac{1}{(\,in\omega_0)^2}\left(e^{-\,in\omega_0 T} - 1\right)
$$

Hence

$$
c_n = \frac{A}{T^2}\left(\frac{T}{-\,in\omega_0}\right) = \frac{\text{i}A}{2\pi n} \qquad n = \pm 1, \pm 2, \ldots
$$

Note that

$$
c_{-n} = \frac{\text{i}A}{2\pi(-n)} = \frac{-\,\text{i}A}{2\pi n} = c_n^* \qquad \text{as it must}
$$

Also $c_0 = \dfrac{1}{T}\displaystyle\int_0^T \dfrac{At}{T}\,dt = \dfrac{A}{2}$ as expected.

Hence the required complex Fourier series is

$$
f(t) = \frac{A}{2} + \frac{\text{i}A}{2\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{\,in\omega_0 t}}{n}
$$

which could be written, showing only the constant and the first two harmonics, as

$$
f(t) = \frac{A}{2\pi}\left\{\ldots - \text{i}\frac{e^{-\,i2\omega_0 t}}{2} - \text{i}e^{-\,i\omega_0 t} + \pi + \text{i}e^{\,i\omega_0 t} + \text{i}\frac{e^{\,i2\omega_0 t}}{2} + \ldots\right\}.
$$

The corresponding trigonometric Fourier series for the function can be readily obtained from this complex series by combining the terms in $\pm n$, $n = 1, 2, 3, \ldots$

For example this first harmonic is

$$
\frac{A}{2\pi}\left\{-\,\text{i}e^{-\,i\omega_0 t} + \text{i}e^{\,i\omega_0 t}\right\} = \frac{A}{2\pi}\left\{-\,\text{i}(\cos\omega_0 t - \text{i}\sin\omega_0 t) + \text{i}(\cos\omega_0 t + \text{i}\sin\omega_0 t)\right\}
$$

$$
= \frac{A}{2\pi}(-2\sin\omega_0 t) = -\frac{A}{\pi}\sin\omega_0 t
$$

Performing similar calculations on the other harmonics we obtain the trigonometric form of the Fourier series

$$
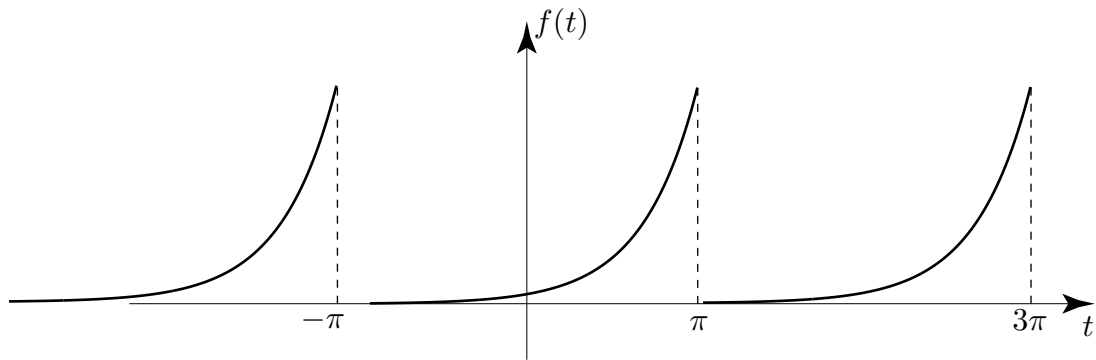f(t) = \frac{A}{2} - \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega_0 t}{n}.
$$

Task

Find the complex Fourier series of the periodic function:

$$f(t) = e^t \qquad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t)$$

Firstly write down an integral expression for the Fourier coefficients c_n:

Your solution

Answer
We have, since $T = 2\pi$, so $\omega_0 = 1$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^t e^{-int} dt$$

Now combine the real exponential and the complex exponential as one term and carry out the integration:

Your solution

Answer
We have

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-in)t} dt = \frac{1}{2\pi} \left[\frac{e^{(1-in)t}}{(1-in)} \right]_{-\pi}^{\pi} = \frac{1}{2\pi} \frac{1}{(1-in)} \left(e^{(1-in)\pi} - e^{-(1-in)\pi} \right)$$

Now simplify this as far as possible and write out the Fourier series:

Your solution

Answer

$$e^{(1-\text{i}n)\pi} = e^{\pi}\, e^{-\text{i}n\pi} = e^{\pi}(\cos n\pi - \text{i}\sin n\pi) = e^{\pi}\cos n\pi$$

$$e^{-(1-\text{i}n)\pi} = e^{-\pi}e^{\text{i}n\pi} = e^{-\pi}\cos n\pi$$

Hence $\quad c_n = \dfrac{1}{2\pi}\dfrac{1}{(1-\text{i}n)}(e^{\pi} - e^{-\pi})\cos n\pi = \dfrac{\sinh\pi}{\pi}\dfrac{(1+\text{i}n)}{(1+n^2)}\cos n\pi$

Note that the coefficients c_n $n = \pm1, \pm2, \ldots$ have both real and imaginary parts in this case as the function being expanded is neither even nor odd.

Also $\quad c_{-n} = \dfrac{\sinh\pi}{\pi}\dfrac{(1-\text{i}n)}{(1+(-n)^2)}\cos(-n\pi) = \dfrac{\sinh\pi}{\pi}\dfrac{(1-\text{i}n)}{(1+n^2)}\cos n\pi = c_n^{*}$ as required.

This includes the constant term $c_0 = \dfrac{\sinh\pi}{\pi}$. Hence the required Fourier series is

$$f(t) = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\frac{(1+\text{i}n)}{(1+n^2)}e^{\text{i}nt} \qquad \text{since} \quad \cos n\pi = (-1)^n.$$

4. Parseval's theorem

This is essentially a mathematical theorem but has, as we shall see, an important engineering interpretation particularly in electrical engineering. Parseval's theorem states that if $f(t)$ is a periodic function with period T and if c_n $(n = 0, \pm 1, \pm 2, \dots)$ denote the complex Fourier coefficients of $f(t)$, then

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

In words the theorem states that the mean square value of the signal $f(t)$ over one period equals the sum of the squared magnitudes of all the complex Fourier coefficients.

Proof of Parseval's theorem.

Assume $f(t)$ has a complex Fourier series of the usual form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T} \right)$$

where

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega_0 t}\, dt$$

Then

$$f^2(t) = f(t)f(t) = f(t) \sum c_n e^{in\omega_0 t} = \sum c_n f(t) e^{in\omega_0 t}$$

Hence

$$
\begin{aligned}
\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\, dt &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum c_n f(t) e^{in\omega_0 t}\, dt \\
&= \frac{1}{T} \sum c_n \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{in\omega_0 t}\, dt \\
&= \sum c_n c_n^* \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}
$$

which completes the proof.

Parseval's theorem can also be written in terms of the Fourier coefficients a_n, b_n of the trigonometric Fourier series. Recall that

$$c_0 = \frac{a_0}{2} \qquad c_n = \frac{a_n - ib_n}{2} \quad n = 1, 2, 3, \dots \qquad\qquad c_n = \frac{a_n + ib_n}{2} \quad n = -1, -2, -3, \dots$$

so

$$|c_n|^2 = \frac{a_n^2 + b_n^2}{4} \qquad n = \pm 1, \pm 2, \pm 3, \dots$$

so

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{a_0^2}{4} + 2\sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{4}$$

and hence Parseval's theorem becomes

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{7}$$

The engineering interpretation of this theorem is as follows. Suppose $f(t)$ denotes an electrical signal (current or voltage), then from elementary circuit theory $f^2(t)$ is the instantaneous power (in a 1 ohm resistor) so that

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t)\,dt$$

is the energy dissipated in the resistor during one period.
Now a sinusoid wave of the form

$$A\cos\omega t \qquad (\text{ or } A\sin\omega t)$$

has a mean square value $\dfrac{A^2}{2}$ so a purely sinusoidal signal would dissipate a power $\dfrac{A^2}{2}$ in a 1 ohm resistor. Hence Parseval's theorem in the form (7) states that the average power dissipated over 1 period equals the sum of the powers of the constant (or d.c.) components and of all the sinusoidal (or alternating) components.

Task

The triangular signal shown below has trigonometric Fourier series

$$f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1 \\ (\text{ odd } n)}}^{\infty} \frac{\cos nt}{n^2}.$$

[This was deduced in the Task in Section 23.3, page 39.]
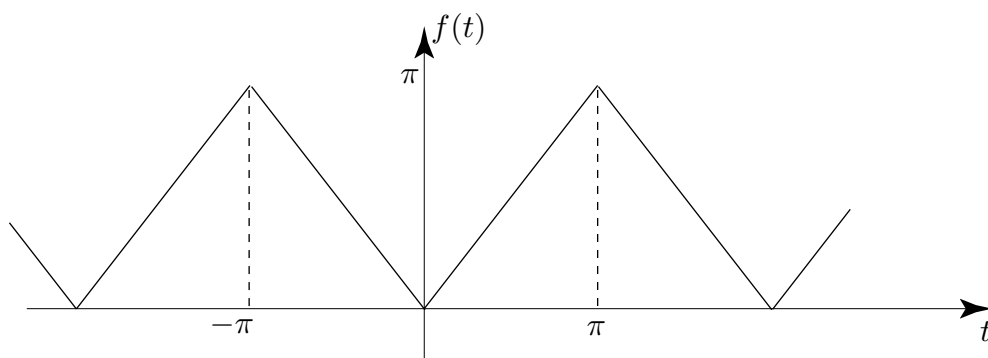
Use Parseval's theorem to show that $\displaystyle\sum_{\substack{n=1 \\ (n \ \ \text{odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}$.

First, identify a_0, a_n and b_n for this situation and write down the definition of $f(t)$ for this case:

Your solution

Answer

We have $\dfrac{a_0}{2} = \dfrac{\pi}{2}$

$$
a_n = \begin{cases} -\dfrac{4}{n^2\pi} & n = 1, 3, 5, \ldots \\[2mm] 0 & n = 2, 4, 6, \ldots \end{cases}
$$

$b_n = 0 \qquad n = 1, 2, 3, 4, \ldots$

Also

$$f(t) = |t| \qquad -\pi < t < \pi$$

$$f(t + 2\pi) = f(t)$$

Now evaluate the integral on the left hand side of Parseval's theorem and hence complete the problem:

Your solution

Answer

We have $f^2(t) = t^2$ so

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(t) \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{1}{2\pi} \left[\frac{t^3}{3} \right]_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

The right-hand side of Parseval's theorem is

$$\frac{a_0^2}{4} + \sum_{n=1}^{\infty} a_n^2 = \frac{\pi^2}{4} + \frac{1}{2} \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{16}{n^4 \pi^2}$$

Hence

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2} \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} \qquad \therefore \qquad \frac{8}{\pi^2} \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^2}{12} \qquad \therefore \qquad \sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{96}.$$

Exercises

Obtain the complex Fourier series for each of the following functions of period 2π.

1. $f(t) = t \qquad -\pi \le t \le \pi$

2. $f(t) = t \qquad 0 \le t \le 2\pi$

3. $f(t) = e^t \qquad -\pi \le t \le \pi$

Answers

1. $\text{i} \sum \dfrac{(-1)^n}{n} e^{int}$ (sum from $-\infty$ to ∞ excluding $n = 0$).

2. $\pi + \text{i} \sum \dfrac{1}{n} e^{int}$ (sum from $-\infty$ to ∞ excluding $n = 0$).

3. $\dfrac{\sinh \pi}{\pi} \sum (-1)^n \dfrac{(1 + \text{i}n)}{(1 + n^2)} e^{int}$ (sum from $-\infty$ to ∞).

An Application of Fourier Series

Introduction

In this Section we look at a typical application of Fourier series. The problem we study is that of a differential equation with a periodic (but non-sinusoidal) forcing function. The differential equation chosen models a lightly damped vibrating system.

Prerequisites Before starting this Section you should ...	• know how to obtain a Fourier series • be competent to use complex numbers • be familiar with the relation between the exponential function and the trigonometric functions
Learning Outcomes On completion you should be able to ...	• solve a linear differential equation with a periodic forcing function using Fourier series

1. Modelling vibration by differential equation

Vibration problems are often modelled by ordinary differential equations with constant coefficients. For example the motion of a spring with stiffness k and damping constant c is modelled by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \tag{1}$$

where $y(t)$ is the displacement of a mass m connected to the spring. It is well-known that if $c^2 < 4mk$, usually referred to as the lightly damped case, then

$$y(t) = e^{-\alpha t}(A\cos\omega t + B\sin\omega t) \tag{2}$$

i.e. the motion is sinusoidal but damped by the negative exponential term. In (2) we have used the notation

$$\alpha = \frac{c}{2m} \qquad \omega = \frac{1}{2m}\sqrt{4km - c^2} \quad \text{to simplify the equation.}$$

The values of A and B depend upon initial conditions.

The system represented by (1), whose solution is (2), is referred to as an **unforced damped harmonic oscillator**.

A lightly damped oscillator driven by a time-dependent forcing function $F(t)$ is modelled by the differential equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \tag{3}$$

The solution or **system response** in (3) has two parts:

    (a) A **transient** solution of the form (2),

    (b) A **forced** or **steady state** solution whose form, of course, depends on $F(t)$.

If $F(t)$ is sinusoidal such that

$$F(t) = A\sin(\Omega t + \phi) \quad \text{where } \Omega \text{ and } \phi \text{ are constants,}$$

then the steady state solution is fairly readily obtained by standard techniques for solving differential equations. If $F(t)$ is periodic but non-sinusoidal then Fourier series may be used to obtain the steady state solution. The method is based on the **principle of superposition** which is actually applicable to any linear (homogeneous) differential equation. (Another engineering application is the series LCR circuit with an applied periodic voltage.)

The principle of superposition is easily demonstrated:-
Let $y_1(t)$ and $y_2(t)$ be the steady state solutions of (3) when $F(t) = F_1(t)$ and $F(t) = F_2(t)$ respectively. Then

$$m\frac{d^2y_1}{dt^2} + c\frac{dy_1}{dt} + ky_1 = F_1(t)$$

$$m\frac{d^2y_2}{dt^2} + c\frac{dy_2}{dt} + ky_2 = F_2(t)$$

Simply adding these equations we obtain

$$m\frac{d^2}{dt^2}(y_1 + y_2) + c\frac{d}{dt}(y_1 + y_2) + k(y_1 + y_2) = F_1(t) + F_2(t)$$

from which it follows that if $F(t) = F_1(t) + F_2(t)$ then the system response is the sum $y_1(t) + y_2(t)$. This, in its simplest form, is the principle of superposition. More generally if the forcing function is

$$F(t) = \sum_{n=1}^{N} F_n(t)$$

then the response is $y(t) = \sum_{n=1}^{N} y_n(t)$ where $y_n(t)$ is the response to the forcing function $F_n(t)$.

Returning to the specific case where $F(t)$ is periodic, the solution procedure for the steady state response is as follows:

Step 1: Obtain the Fourier series of $F(t)$.

Step 2: Solve the differential equation (3) for the response $y_n(t)$ corresponding to the nth harmonic in the Fourier series. (The response y_o to the constant term, if any, in the Fourier series may have to be obtained separately.)

Step 3: Superpose the solutions obtained to give the overall steady state motion:

$$y(t) = y_0(t) + \sum_{n=1}^{N} y_n(t)$$

The procedure can be lengthy but the solution is of great engineering interest because if the frequency of one harmonic in the Fourier series is close to the **natural frequency** $\sqrt{\dfrac{k}{m}}$ of the undamped system then the response to that harmonic will dominate the solution.

# 2. Applying Fourier series to solve a differential equation

The following Task which is quite long will provide useful practice in applying Fourier series to a practical problem. Essentially you should follow Steps 1 to 3 above carefully.

Task

The problem is to find the steady state response $y(t)$ of a spring/mass/damper system modelled by

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F(t) \tag{4}$$

where $F(t)$ is the **periodic square wave** function shown in the diagram.
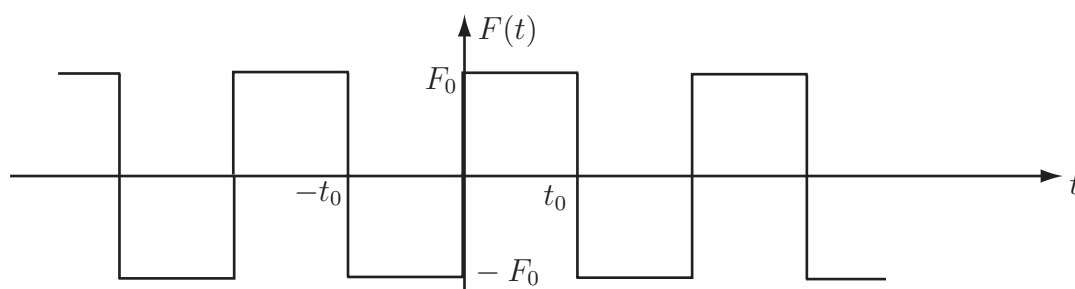
Step 1: Obtain the Fourier series of $F(t)$ noting that it is an odd function:

Your solution

Answer

The calculation is similar to those you have performed earlier in this Workbook.

Since $F(t)$ is an odd function and has period $2t_0$ so that $\omega = \dfrac{2\pi}{2t_0} = \dfrac{\pi}{t_0}$, it has Fourier coefficients:

$$
\begin{aligned}
b_n &= \frac{2}{t_0} \int_0^{t_o} F_0 \sin\left(\frac{n\pi t}{t_0}\right) dt \qquad n = 1, 2, 3, \ldots \\
&= \left(\frac{2F_0}{t_0}\right)\left(\frac{t_0}{n\pi}\right)\left[-\cos\frac{n\pi t}{t_0}\right]_0^{t_0} \\
&= \frac{2F_0}{n\pi}(1 - \cos n\pi) = \begin{cases} \dfrac{4F_0}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}
$$

so $\quad F(t) = \dfrac{4F_0}{\pi} \displaystyle\sum_{n=1}^{\infty} \dfrac{\sin n\omega t}{n} \qquad$ (where the sum is over odd n only).

Step 2(a):

Since each term in the Fourier series is a sine term you must now solve (4) to find the steady state response y_n to the nth harmonic input: $\quad F_n(t) = b_n \sin n\omega t \qquad n = 1, 3, 5, \ldots$

From the basic theory of linear differential equations this response has the form

$$ y_n = A_n \cos n\omega t + B_n \sin n\omega t \tag{5} $$

where A_n and B_n are coefficients to be determined by substituting (5) into (4) with $F(t) = F_n(t)$. Do this to obtain simultaneous equations for A_n and B_n:

Your solution

Answer

We have, differentiating (5),

$$
\begin{aligned}
y_n' &= n\omega(-A_n \sin n\omega t + B_n \cos n\omega t) \\
y_n'' &= (n\omega)^2(-A_n \cos n\omega t - B_n \sin n\omega t)
\end{aligned}
$$

from which, substituting into (4) and collecting terms in $\cos n\omega t$ and $\sin n\omega t$,

$$(-m(n\omega)^2 A_n + cn\omega B_n + kA_n)\cos n\omega t + (-m(n\omega)^2 B_n - cn\omega A_n + kB_n)\sin n\omega t = b_n \sin n\omega t$$

Then, by comparing coefficients of $\cos n\omega t$ and $\sin n\omega t$, we obtain the simultaneous equations:

$$(k - m(n\omega)^2)A_n + c(n\omega)B_n = 0 \tag{6}$$

$$-c(n\omega)A_n + (k - m(n\omega)^2)B_n = b_n \tag{7}$$

Step 2(b):

Now solve (6) and (7) to obtain A_n and B_n:

Your solution

Answer

$$A_n = -\frac{c\omega_n b_n}{(k - m\omega_n^2)^2 + \omega_n^2 c^2} \qquad (8)$$

$$B_n = \frac{(k - m\omega_n^2)b_n}{(k - m\omega_n^2)^2 + \omega_n^2 c^2} \qquad (9)$$

where we have written ω_n for $n\omega$ as the frequency of the nth harmonic

It follows that the steady state response y_n to the nth harmonic of the Fourier series of the forcing function is given by (5). The amplitudes A_n and B_n are given by (8) and (9) respectively in terms of the systems parameters k, c, m, the frequency ω_n of the harmonic and its amplitude b_n. In practice it is more convenient to represent y_n in the so-called **amplitude/phase** form:

$$y_n = C_n \sin(\omega_n t + \phi_n) \qquad (10)$$

where, from (5) and (10),

$$A_n \cos \omega_n t + B_n \sin \omega_n t = C_n(\cos \phi_n \sin \omega_n t + \sin \phi_n \cos \omega_n t).$$

Hence

$$C_n \sin \phi_n = A_n \qquad C_n \cos \phi_n = B_n$$

so

$$\tan \phi_n = \frac{A_n}{B_n} = \frac{c\omega_n}{(m\omega_n^2 - k)^2} \qquad (11)$$

$$C_n = \sqrt{A_n^2 + B_n^2} = \frac{b_n}{\sqrt{(m\omega_n^2 - k)^2 + \omega_n^2 c^2}} \qquad (12)$$

Step 3:

Finally, use the superposition principle, to state the complete steady state response of the system to the periodic square wave forcing function:

Your solution

Answer

$$y(t) = \sum_{n=1}^{\infty} y_n(t) = \sum_{\substack{n=1 \\ (n \text{ odd})}} C_n(\sin \omega_n t + \phi_n) \quad \text{where } C_n \text{ and } \phi_n \text{ are given by (11) and (12)}.$$

In practice, since $b_n = \frac{4F_0}{n\pi}$ it follows that the amplitude C_n also decreases as $\frac{1}{n}$. However, if one of the harmonic frequencies say ω_n' is close to the natural frequency $\sqrt{\frac{k}{m}}$ of the undamped oscillator then that particular frequency harmonic will dominate in the steady state response. The particular value ω_n' will, of course, depend on the values of the system parameters k and m.